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PROCEEDINGS
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ASYMMETRY OF LIGHT CURVES IN THE GREAT SEQUENCE AS A FUNCTION OF FREQUENCY OF PERIOD

By Sergei Gaposchkin

Harvard College Observatory

Communicated December 9, 1937

The cluster-type variables, the Cepheids, and the long-period variables are generally understood to belong to one large group of stars in which the variations of light, spectrum and radial velocity are considered to have intrinsically the same cause. This group of variable stars may be conveniently called the Great Sequence. The so-called semi-regular variables like RV Tauri are undoubtedly related to the group. All the available light curves of stars of the Great Sequence have recently been investigated by Mrs. Gaposchkin and the writer, and one of the more significant results of the survey is presented in this paper. The complete results will be published elsewhere. There have been previous investigations concerning the relation between the period and the shape of the light curve. Detailed references for long-period variables are given by Ludendorff (*Hand. d. Ap.*, **6**, 92, 1928) and Thomas (*Veröff. Ber.-Bab.*, **9**, No. 4, 1932), and for Cepheids, by Parenago and Kukarkin (*Zeits. f. Ap.*, **11**, 337, 1937).

Differences in shape of light curve suggest classifications. It is characteristic of the Great Sequence that similar light curves are found in all the constituent groups of stars, although there are large differences within any one group. There have been useful attempts, such as those of Ludendorff, Campbell, Thomas, Phillips, Hertzsprung, Parenago and Kukarkin, to classify or describe the variety of light curves, but all of these classifications have been confined to some one single subgroup. We are here concerned with the period–light curve relationships throughout the entire Great Sequence. For this purpose we have selected three parameters to describe the light curve: the ratios a/b, c/d and e/f, where a and b are two areas designated in figure 1a; c and d are two areas designated in figure 1b. The horizontal line has been drawn through the median magnitudes; e is the distance from the "bump" to the following minimum, and f is the distance from the maximum to the minimum. The three ratios furnish a complete description of the light curve, permitting the expression of the

following features: asymmetry with respect to the ordinate (speed of rise and fall); asymmetry with respect to the abscissa (duration of maximum and minimum); existence of a bump or shoulder on the ascending or descending branch. Our ratio a/b is closely correlated with the quantity $(m - M)/P$, which, with some modification, has been extensively used to characterize the light curves of Cepheids, or even long-period variables.

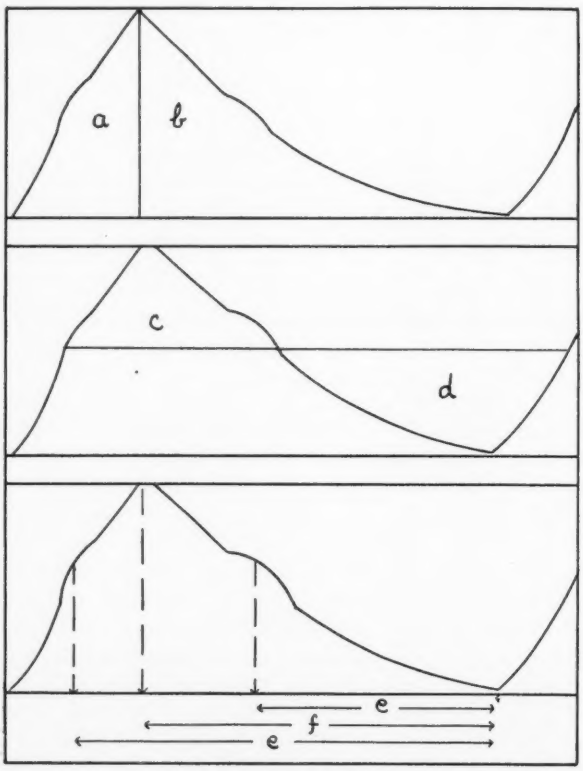


FIGURE 1
Parameters for describing light curve: a/b, c/d and e/f

But $(m - M)/P$ is a less precise and elastic quantity than a/b, especially for light curves with a more or less constant brightness at minimum, for which the position of the actual minimum is indefinite, and the value of $(m - M)/P$ correspondingly doubtful. The ratio a/b is less sensitive to this source of uncertainty.

The general character of the three ratios is illustrated below:

$a/b = 1.0$	Rise and fall equally steep
$a/b < 1.0$	Rise steeper than fall
$a/b > 1.0$	Rise less steep than fall
$c/d = 1.0$	Maximum and minimum equal
$c/d < 1.0$	Maximum shorter (sharper) than minimum
$c/d > 1.0$	Maximum longer (less sharp) than minimum
$e/f < 1.0$	Bump on descending branch
$e/f > 1.0$	Bump on rising branch

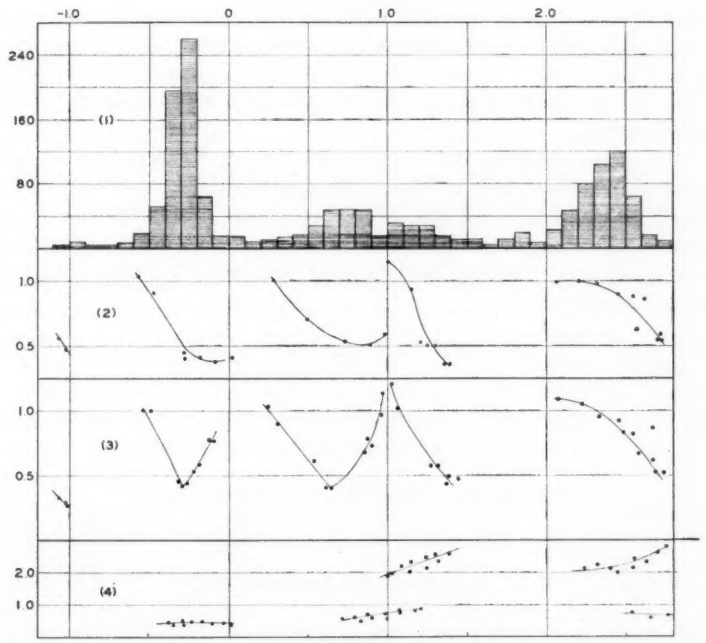


FIGURE 2

Relation of light curve to period in the Great Sequence. Abscissae, logarithms of period; ordinates, (1) the frequency of period, (2) the parameter a/b, (3) the parameter c/d, (4) the parameter e/f.

About a thousand stars belonging to the Great Sequence, including a number of variables in globular clusters, have been investigated. Within each subgroup of the Great Sequence there is large variation in the values of the three ratios. Most of the light curves show some asymmetry. The results shown in figure 2 may be summarized as follows:

(1) There are several maxima and minima of period-frequency. Maxima lie at about $0^d.1$, $0^d.5$, $5^d.0$, $10^d.1$, $85^d.0$, $320^d.0$, the first corresponding to cluster-type variables, and the sixth to the long-period variables, which probably, themselves, constitute a double group. The minima lie at about $0^d.2$, $1^d.6$, $9^d.0$ and 50–$60^d.0$, and possibly at about 600^d.

(2) Between any two minima of frequency the light curves run, with increasing period, from symmetry to asymmetry.

(3) The long-period Cepheids, and the long-period variables of longest period, show bumps on the rising branch of the light curve, which are nearest to the minimum for the longest periods. The bumps on the descending branch seem to occur at about the same phase, whatever the period. The cluster-type Cepheids do not show bumps on the rising branch of the light curve.

(4) At the minima of period-frequency the light curves show the greatest irregularities.

It may be remarked that there is no definite indication of a period-spectrum relation within the Great Sequence similar to the relations mentioned above, except perhaps at the minimum near sixty days. It is probable that the groups of stars are not mutually exclusive in period, but overlap slightly; with the two groups of long-period variables the overlap is considerable.

NOTE ON THE STELLAR DISTRIBUTION IN THE VICINITY OF A SOUTHERN GALACTIC WINDOW

By Bart J. Bok and Eric M. Lindsay

Harvard College Observatory

Communicated December 15, 1937

Harvard and Mount Wilson studies on the distribution of faint external galaxies have led to the discovery of several galactic "windows," regions in comparatively low latitudes which are unusually rich in distant extra-galactic nebulae. For most fields within fifteen degrees of the galactic equator, the total absorption of star light in our Milky Way system is so large that very few nebulae are found on a plate. The discovery of a galactic window is by itself excellent evidence for the smallness of the total galactic absorption in that particular direction. The present note deals with the stellar distribution for the most conspicuous southern galactic window, discovered by Shapley, which is located near galactic longitude $\lambda = 300°$, between galactic latitudes $\beta = -10°$ and $-16°$. An estimate

of the total galactic absorption from the observed frequency of external galaxies gives $\Delta m = 0.4$ for photographic light.

1. *Star Counts.*—Shapley and Miss Boyd[1] have published star counts made by Miss R. Jones from a Bruce plate centered at $\alpha = 16^h52^m.3$, $\delta = -62° 28'$. The regions at and around the galactic window are, moreover, in the part of the sky covered by Lindsay's unpublished star counts[2] between $m_{pg} = 10.0$ and 13.5, for the southern galactic belt. Since it will probably be some time before the analysis of Lindsay's complete material will be ready for publication, it appeared of interest to make a preliminary study of the counts for the region in the vicinity of this "window."

Lindsay's star counts suggest the following values of log $N(m)$ for the center of the Bruce plate counted by Miss Jones:

m_{pg}:	10.0	11.0	12.0	13.0	13.5
log $N(m)$:	1.00	1.35	1.72	2.00	2.25

The counts by Miss Jones gave the average values:

m_{pg}:	14.0	15.0	16.0	17.0	18.2
log $N(m)$:	2.47	3.01	3.31	3.84	4.19

$N(m)$ represents in both cases the number of stars of apparent photographic magnitude m or brighter per square degree. Let $A(m)$ represent the corresponding number of apparent magnitude m, defined by the formula $A(m) = dN(m)/dm$. We have assumed in our calculations that the smoothed values of $A(m)$ of table 1 are representative for the southern galactic window.

TABLE 1

STAR COUNTS FOR THE SOUTHERN GALACTIC WINDOW

m_{pg}	$A(m)$	m_{pg}	$A(m)$
10.0	7.7	15.0	965
11.0	17.4	16.0	2430
12.0	41.0	17.0	5780
13.0	103.0	18.0	11900
14.0	330.0		

Lindsay's star counts for neighboring fields show that the star counts to $m = 13.5$ do not indicate the presence of an excess of stars for the region of the galactic window. The curves for the variation of log $N(m)$ with the galactic longitude for various zones of galactic latitude show no conspicuous variations between $\lambda = 280°$ and $\lambda = 320°$ for the zones $\beta = -8°$ to $-12°$ and $\beta = -13°$ to $-17°$. The star counts prove that between $m = 10.0$ and 13.5 the values of log $N(m)$ in the galactic window do not differ by more than ± 0.10 from those for the regions outside the window at the same latitude. Lindsay's counts were made from plates

taken for the purpose with the 8-inch Bache refractor, and great care was taken to obtain counts on a homogeneous photometric system. We feel justified therefore, to conclude that *the numbers of stars brighter than* $m_{pg} = 13.5$ *for the region of the galactic window do not differ significantly from those of neighboring fields at the same galactic latitudes.*

Comparable data are not yet available for the fainter magnitudes. A comparison between Miss Jones's counts and average numbers of stars, published by van Rhijn,[3] shows the region of the galactic window to have fewer stars than nearby comparison regions at the same latitude. The photometric standards for faint southern stars are very uncertain, but the comparison renders it extremely unlikely that a significant excess in the window is present at $m = 18.0$.

The absence of an appreciable excess of stars brighter than the fifteenth magnitude is confirmed by an inspection of plates of one to two hours' exposure taken with short focus cameras whose apertures vary between one and three inches. The stellar distribution for the region between $\lambda = 280°$ and $320°$, and $\beta = -10°$ to $-20°$ is apparently quite regular.

2. *Absorption Problems.*—It is of interest to determine the maximum permissible value of the total galactic absorption for photographic light for the regions ten to fifteen degrees on either side of the galactic window but at the same galactic latitude. Since there is no evidence to the contrary, we may assume that the same distribution of star density will hold for the direction of the galactic window as for the comparison regions on either side. The run of the space densities for the direction of the galactic window may be determined from the star counts of table 1. The total absorption for photographic light for the direction of the galactic window is of the order of $0^m.4$. It makes little difference for the analysis how this small amount of absorption is distributed along the line of sight; we have assumed a uniform absorption of $0^m.1$ per 100 parsecs over the first 400 parsecs. The run of the space densities for the direction of the galactic window is shown in table 2.

TABLE 2

DISTRIBUTION OF SPACE DENSITIES FOR GALACTIC WINDOW

r	z	$D(r)$	r	z	$D(r)$
100	20	1.00	1600	320	0.20
160	32	0.85	2500	500	0.15
250	50	0.70	4000	800	0.11
400	80	0.55	6300	1260	0.08
630	126	0.40	10000	2000	0.05
1000	200	0.30			

In table 2, r is the distance in the direction of the galactic window, and $D(r)$ the corresponding space density. The latitude of the region for

which the star counts are available is $\beta = -13°$ on the Harvard pole, and $\beta = -10°$ on van Rhijn's pole; the value $\beta = -11°.5$ for which $\sin \beta = 0.20$ has been assumed for the average latitude of the counted region. The relation between z, the distance above the galactic plane, and the corresponding value of r is then simply: $z = r/5$. The change of luminosity function with increasing distance from the galactic plane has been taken into account;[4] the tabulated star densities refer to the stars with $M = 0$ to $+5$.

It is now a simple matter to predict the run of the log $A(m)$ values for one of the comparison regions at $\beta = -11°.5$ in which very few faint galaxies are observed, on various assumptions as to the total absorption for photographic light in the comparison regions. The computations have been carried through for three values of the total absorption, $\Delta m = 1^m.0$, $\Delta m = 2^m.0$ and $\Delta m = 3^m.0$. For lack of better information we shall assume in each case that the absorption is uniformly distributed over the interval $0 < r < 1500$ parsecs and that no absorption occurs beyond $r = 1500$ parsecs. The exact way in which the absorption is distributed along the line of sight is immaterial for the following arguments.

The predicted values of log $A(m)$ will be found in table 3 and again in graphical form in the accompanying diagram; the log $A(m)$'s for the region of the galactic window have been given for comparison.

TABLE 3

PREDICTED VALUES OF LOG $A(m)$ FOR THREE OBSCURED COMPARISON REGIONS

m	GALAC. WINDOW (OBSERVED)	$\Delta m = 1^m.00$	$\Delta m = 2^m.00$ (PREDICTED)	$\Delta m = 3^m.00$
10.0	0.89	0.80	0.73	0.65
11.0	1.24	1.11	1.00	0.89
12.0	1.61	1.50	1.32	1.19
13.0	2.01	1.90	1.65	1.45
14.0	2.52	2.29	1.98	1.76
15.0	2.98	2.69	2.34	2.04
16.0	3.39	3.11	2.72	2.37
17.0	3.76	3.49	3.11	2.67
18.0	4.07	3.87	3.50	3.12

Lindsay's counts for $m_{pg} = 10.0$ to 13.5 showed no indications of the presence of an excess of stars in the galactic window. It was pointed out that an inequality of the order of ± 0.10 in the log $N(m)$'s—and presumably a similar variation in the log $A(m)$'s—might have escaped detection. A comparison between the log $A(m)$'s in the second column with those in the fourth and fifth columns shows that the cases $\Delta m = 2^m.00$ and $\Delta m = 3^m.00$ are definitely out of the picture. The somewhat less accurate star counts for $m_{pg} = 14$ to 18 fully confirm this conclusion: either the photographic magnitudes used by Miss Jones or those used by

van Rhijn would have to be in error by more than a magnitude at m_{pg} = 18.0, if the total photographic absorption in the comparison regions were as large as $\Delta m = 2^m.00$.

The available evidence on stellar distribution shows that *the total photographic absorption for the regions at* $\beta = -10°$ *to* $-15°$ *between* $\lambda = 280°$ *and* $\lambda = 320°$ *does not exceed* $\Delta m = 1^m.50$, *and that a more probable value is* $\Delta m \leqq 1^m.00$. A total absorption $\Delta m = 1^m.00$ for a region at $\beta = -11°.5$ implies that the half-value for what has commonly been designated as the

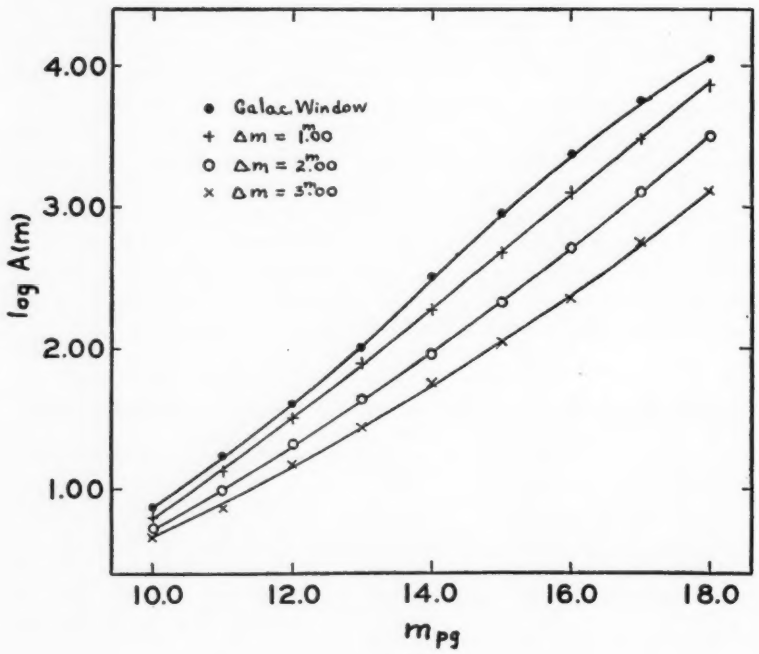


FIGURE 1

"optical thickness" of the galactic absorbing layer is of the order of $0^m.20$ for this particular part of the sky. It is doubtful whether the term "optical thickness" means very much for an absorbing layer that is apparently far from homogeneous, but the value $0^m.20$ may have some significance when applied as an average for a region covering at least 40° in galactic longitude, for which no conspicuous variations in the stellar distribution have been observed.

No certain conclusions as to the amount of the average coefficient of absorption per unit distance *in* the galactic plane can be derived from ob-

servations at $\beta = -11^{\circ}.5$. The galactic absorption is probably largely confined to a layer with a half-thickness of 200 to 400 parsecs. Our results render it unlikely that the average coefficient of absorption for photographic light near the galactic plane will exceed $1^{m}.0$ per kiloparsec.

3. *Suggestions for Future Work.*—It is important that the suggested value $\Delta m \leqq 1^{m}.00$ for the entire region $\beta = -10^{\circ}$ to -15° and $\lambda = 280^{\circ}$ to 320° be tested. The stellar distribution for this part of the Milky Way seems to be regular. The location of the region under discussion between the center of the greater galactic system and the possible center of a local system makes this part of the sky one of the most interesting for galactic research. Fortunately there is here no evidence for an extended dark nebula similar to the Ophiuchus nebula, which distorts the true features of galactic structure between longitudes $\lambda = 300^{\circ}$ and 340° north of the galactic plane.

If future studies on the distribution of faint external galaxies give a value $\Delta m > 1^{m}.50$ for the regions on either side of the galactic window, it will probably be necessary to explain the abundance of galaxies in the window by a metagalactic cloud of distant nebulae. It appears most unlikely that the star counts on which our discussion is based would be so much in error as to alter the broad conclusions of the present note. The only remaining alternative would be to account for the galactic absorption by a peripheral absorbing medium which would only become effective at a distance of at least 1000 parsecs below the galactic plane; such a hypothesis cannot be accepted unless supported by strong independent evidence.

The absorption and structural properties of the region as a whole will become better known if studies on the distribution of faint external galaxies go hand in hand with studies on the distribution of colors and variable stars, spectral surveys and star counts to faint magnitude limits on a definite photometric system. For several years our ignorance concerning galactic absorption has appeared to be the chief obstacle in the way of effective analysis of galactic structure; this obstacle can now be removed. Our knowledge of galactic structure can be further advanced if detailed studies of the distribution of stars and nebulae are made for more or less obscured fields near galactic windows as well as for the windows themselves.

[1] *Harvard Annals*, **105**, 252 (1937).
[2] Bok, *The Distribution of the Stars in Space*, 59 (1937).
[3] *Groningen Publications* 43 (1929).
[4] *Zeit. f. Ap.*, **10**, 161 (1935).

THE PRESENT STATE OF OUR KNOWLEDGE CONCERNING THE LIFE CYCLE OF THE FORAMINIFERA

BY EARL H. MYERS

COMPTON JUNIOR COLLEGE, COMPTON, CALIF.

Communicated November 20, 1937

The recent extensive use of Foraminifera in solving problems of stratigraphic geology and the recognition of their value as guide fossils in petroleum geology has greatly stimulated interest in the biological problems of these marine rhizopods, consequently many investigators are now contributing to our knowledge of various features of these organisms. Recent contributions include studies of the cytological behavior of the nuclei in reproduction, the rate of reproduction, the life span, those factors which limit their geographic and bathymetric distribution in the sea and the interpretation of the stratigraphic distribution of geological populations on the floor of present oceans from core-samples of sediments.

The conclusions set forth in this paper are the result of one phase of the coöperative study of the Foraminifera planned by Dr. T. Wayland Vaughan and continued under the direction of Dr. H. U. Sverdrup at the Scripps Institution of Oceanography at La Jolla.

In the investigation of the life cycle of the Foraminifera Lister (1895) first explained that the occurrence of two distinct types of individuals within a recognized species was due to an alternation of generations. He described in *Polystomella crispa* the asexual origin of megalospheric young from individuals of the larger microspheric generation and presumed that flagellated organisms observed in megalospheric tests were gametes, i.e., zoospores. He further presumed that following the union of these gametes in fertilization, the resulting amoebula gave rise to the asexual or microspheric generation. Many attempts have been made to substantiate these views, but much of the evidence thus far presented has been fragmentary and controversial.

In 1935 the life cycle of the Foraminifera was for the first time brought into agreement with modern biological concepts as they apply to Protozoan cytology. In the life cycle of *Patellina corrugata* (Myers '35) it was abundantly demonstrated that all nuclei were the result of mitotic division, that the nuclei were confined within a nuclear membrane at all times, and that the haploid gametes were amoebulae and not flagellated. The events in this life cycle have been recorded in a motion picture produced under the direction of Dr. C. A. Kofoid, in which forty-one days of reproductive and developmental activity were photographed. The conclusions resulting from this work were further borne out in the life cycle of *Spirillina vivipara* (Myers '36).

There is little doubt that these more recent life cycles have proved confusing, in that they seem to bear slight resemblance to the life cycle of *Polystomella crispa* which has been generally accepted as typical of the Foraminifera, but basically the two are similar. Dimorphism is common to both. This dimorphism is the result of an alternation of generations. In both, the test of the asexually produced megalospheric generation is smaller and contains but a single nucleus, while that of the larger sexually produced microspheric generation contains many nuclei. That there is a difference in the nature of the gametes and a presumptive difference in the behavior of the nuclei in reproduction is incidental to this basic similarity.

Evidence further supporting the idea that there is a relationship between the life cycle of *P. corrugata* amd *S. vivipara* and those previously proposed was studied this summer for the fifth successive season. In this investigation it was found by prolonged observation of living organisms *in situ* and permanent cytological preparations that the sequence of events in the life cycle of certain species of *Discorbis* and *Bulimina* resemble not only that of *Polystomella crispa* proposed by Lister (1895) and Schaudinn (1903) and substantiated in part by Winter in *Peneroplis pertusus* (1907), but also bears a marked resemblance to that of *Spirillina vivipara* and *Patellina corrugata*.

In the life cycle of *P. corrugata* amd *S. vivipara* gametogenesis, fertilization and the development of the juvenile stage of the asexual generation takes place within a cyst formed about two or more megalospheric individuals associated for the purpose of sexual reproduction. A similar association of individuals, prior to sexual reproduction, occurs in the species of *Discorbis* and *Bulimina* referred to above, but with these modifications. In these species the tests of megalospheric organisms so associated are arranged with the ventral surfaces opposed, while in the former the several tests remain in contact with the substratum and are enclosed within a protective cyst, the gametes are flagellated and not amoeboid, and the juvenile stage of the microspheric generation develops within the common space formed by the dissolution of a portion of the ventral wall and the septa separating the later formed chambers of the associated tests rather than within an enveloping cyst. In *Discorbis opercularis* d'Orbigny and *Discorbis patelliformis* Brady, the diameter of the initial chamber of the sexually produced microspheric generation is not limited by the size of the zygotic amoebula which resulted from the union of two gametes in fertilization, since considerable growth takes place in the zygotic amoebula following fertilization and before the secretion of the microspheric proluculum, the gametes which failed in fertilization serving as food. In *Patellina* and *Spirillina* similar growth activities in the zygotic amoebula prior to test secretion was also observed.. Again we find no evidence in support of the

idea that division of a nucleus of a foraminifer ever results in the production of more than two daughter nuclei.

The dimorphic characters of the respective stages in the life cycle of *Polystomella crispa* amd *Discorbis opercularis* are in close agreement in regard to the relative diameter of the initial chamber of microspheric and megalospheric tests, the number and arrangement of the chambers and the number of nuclei present in each. These characteristics, together with the flagellate nature of the gametes, make it evident that the cytological behavior of the nuclei in reproduction in *Polystomella* and similar dimorphic species should resemble that observed in the species under discussion.

In *D. opercularis* and *D. patelliformis* we have demonstrated for the first time a genetic continuity in the origin of gametic nuclei from the original megalospheric nucleus by an orderly series of nuclear divisions, in a species in which the gametes are flagellated. Further, we have determined from serial sections of more than five hundred pairs of individuals associated for the purpose of sexual reproduction, that fertilization and the development of the zygotic amoebulae into two and three chambered multinucleate microspheric juveniles occur within the brood chamber formed by an associated group of megalospheric tests, thus establishing on demonstrable evidence, both the origin and fate of flagellated gametes in a foraminifer. The association of megalospheric individuals for the purpose of sexual reproduction and the escape of the sexually produced microspheric young from these associated groups of tests have been observed scores of times in cultures maintained in the laboratory.

The occurrence of free flagellated gametes or zoospores derived from megalospheric *Polystomella crispa* has been discussed. Although these observations are in need of verification, we have evidence which tends to support the life cycle proposed in this species by Lister. Since the gametes of *Polystomella* are in all probability free and pelagic, a minimum of opportunity for fertilization is afforded, compared with a state in which the gametes derived from several individuals are confined at all times within a limited area. This difference might explain the occurrence of only one microspheric to more than thirty megalospheric tests in *Polystomella crispa* while in *Discorbis opercularis*, where fertilization and the development of the microspheric generation take place within a brood chamber formed by a group of associated tests, the ratio is less than one to two.

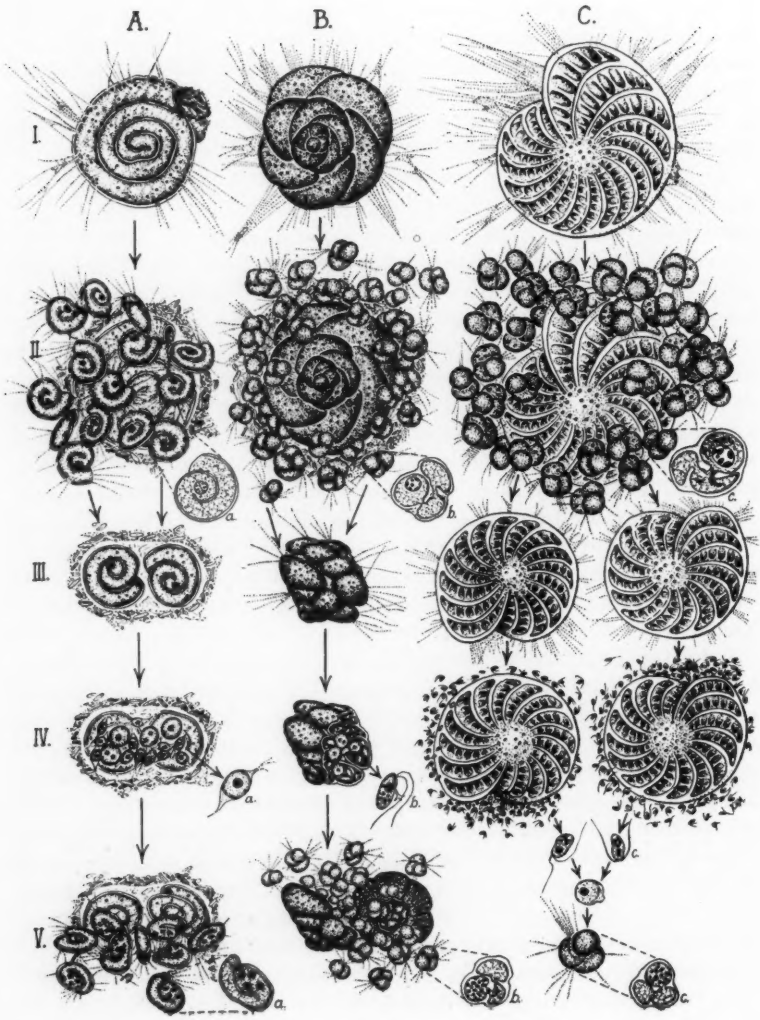
If the gametes, i.e., zoospores, are free and pelagic in those species of foraminifera in which an association of individuals prior to sexual reproduction does not occur, dissemination and distribution may be largely dependent upon this pelagic stage (Vaughan '33).

In *Spirillina vivipara* (Myers '36) there is a reversal in the direction of rotation of the spiral chamber of the tests of the two generations. That a similar condition exists in other species has been reported by Føyn (1936).

This discovery affords an additional characteristic by which one may at times distinguish megalospheric and microspheric tests. Thus we see that in a species such as *Discorbis opercularis* there are several characteristics by which representatives of the respective generations may be identified. In this species the microspheric test contains about forty nuclei, the chambers are usually arranged clockwise in a spire around the initial chamber or proloculum, and this initial chamber is small, while in the megalospheric test only one nucleus is present, the direction of rotation of the chambers is counter-clockwise, and the proloculum is larger.

Three distinct types of tests have been described in certain recognized species of foraminifera. These observations led Hofker (1930) to propose that more than one asexual generation may intervene between two successive sexual generations. Although Hofker's work has met with much criticism, the plan of the life cycle proposed by him is common to other groups of Protozoa and there is no reason why such a life cycle could not occur in the Foraminifera. The existence of such a life cycle is further suggested by the occurrence of many successive asexual generations in isolation cultures of species showing several distinct types of asymmetry. Asexual reproduction of these monoculeate and therefore megalospheric foraminifera has been proven cytologically. Since sexual reproduction is known in other species of the same genus, it would be interesting to determine what would happen if isolation cultures of megalospheric organisms of one of these recognized dimorphic species were to be maintained for an appreciable period after they would have reproduced sexually had the opportunity presented itself.

Polymorphism of a different type has been studied by us in *Tretomphalus bulloides*. The life cycle of this interesting species includes an alternation of generations similar to that described in *Discorbis opercularis* but with this difference. When the asexually produced mononucleate megalospheric bottom dwelling stage reaches maturity, a float chamber containing a spherical gas-filled float is added to the ventral side of the test. If this test is ruptured immediately before it assumes the pelagic phase it is found to contain many thousands of biflagellated gametes and a small amount of residual protoplasm which continues to give rise to pseudopodia after the organism has arrived at the surface. This transformation from a bottom-dwelling to a pelagic stage requires about eighteen hours and its developmental stages have been recorded in series of photographs of living organisms. This pelagic stage is of short duration for all of the gametes are discharged within six hours from the time the test arrives at the surface. When two of these pelagic individuals approach one another they are drawn together by anastomosing pseudopodia so that the large pores in the ventral sides of the float chambers are brought into juxtaposition and the


A.
B.
C.
I.
II.
III.
IV.
V.
a.
b.
c.

gametes immediately begin to pour out in a cloud about the two tests, thus providing an opportunity for cross fertilization. This pelagic stage of *Tretomphalus* is perhaps one of the most remarkable adaptations of a species to assure the wide dissemination of progeny that occurs within the animal kingdom. The bottom-dwelling or megalospheric stage which gives rise to the pelagic *Tretomphalus* bears a marked resemblance to

EXPLANATION OF PLATE

The life cycle of many foraminifera consists of an orderly succession of sexual and asexual phases. This alternation of generations results in test dimorphism. Five stages are diagrammatically shown on Plate I for three species. (A) *Spirillina vivipara*, (B) *Discorbis patelliformis*, and (C) *Polystomella crispa*.

Asexual Generation

Stage I. In *D. patelliformis* and *P. crispa* as in many foraminifera the initial chamber or proloculum of the sexually produced multinucleate agamont is smaller than that of the asexually produced mononucleate gamont and are known as microspheric and megalospheric respectively, while in *S. vivipara*, a more primitive species, there is little difference in the diameter of the proloculum of the two generations. In all species studied, the agamont test is the larger.

Stage II. Following an orderly series of nuclear divisions multiple fission results in as many mononucleate agamonts as there were nuclei produced. In *D. patelliformis* multiple fission and test secretion takes place within the parent test while in *S. vivipara* and *P. crispa* these activities are preceded by the escape of the protoplasmic content from the test. Stage II. (a, b and c) Enlargement of sexual gamont showing the nucleus.

Sexual Generation

Stage III. In *S. vivipara*, sexual reproduction is preceded by the association of gamonts within a cyst composed of substrate debris and animal cement. A somewhat similar association takes place in *D. patelliformis* where no cyst is formed, while in *P. crispa* it is presumed that a close association between gamonts does not occur, the gametes, or zoospores being free and pelagic.

Stage IV. The gametes of *S. vivipara* are amoeboid while those of *D. patelliformis* and *P. crispa* are flagellated. Gametogenesis includes an orderly series of nuclear divisions of the equational type followed by a reduction division. Fertilization takes place between gametes derived from different gamonts. In *S. vivipara* the agamonts develop within a cyst, in *D. patelliformis* within the space formed by the dissolution of septa between the chambers, while in *P. crispa* the gametes are pelagic, fertilization depending upon the chance meeting of gametes. Stage IV. (a, b and c) Enlargement of gametes.

Stage V. Juvenile agamonts are multinucleate as a result of an orderly series of nuclear divisions immediately following fertilization and before the secretion of the test. Agamonts of *S. vivipara* contain but four nuclei while those of *D. patelliformis* and *P. crispa* each have about forty nuclei. In *S. vivipara* the sexually produced young develop within a cyst, those of *D. patelliformis* within the space resulting from the dissolution of the septa between chambers and the ventral surface of associated tests, while those of *P. crispa* develop from the free zygotic amoebula. Stage V. (a, b and c) Enlargement of asexual agamont showing nuclei.

Discorbis globularis. The microspheric stage is larger, more depressed in appearance, and the initial chamber does not approach the diameter of the smallest megalospheric proloculum observed.

Although our observations on *Tretomphalus* extends over a period of five years, some cultures persisting for as long as three years, we have been unable to demonstrate the fate of the gametes beyond their union in fertilization, however, the genetic origin of the gametes has been determined from cytological preparations.

The cytological problem in the study of life cycles of foraminifera is complicated by the presence of many types of cytoplasmic inclusions. Residual chromatin derived from the nuclei of food organisms and other metabolic products which react with stains that are partially specific for chromatin frequently produce confusing results. Symbiotic algae and other organisms contained within the cytoplasm, or occasionally filling all of the chambers of a test, are at times even more confusing. Various flagellates and amoebae, which we have observed, seem in some instances to be limited to certain species. We have also found a nematode worm which during the colder months of the year has an incidence of more than five per cent in *Rotalia turbinata* Cushman and Valentine. Small pieces of flesh of organisms allowed to decompose in sea water provide a substratum for many types of organisms, especially flagellates. Dead foraminifera in cultures frequently present a similar condition and these flagellates might easily be mistaken for gametes.

There is a decided need for further study of life cycles in larger foraminifera and virtually nothing is known concerning the life cycle of those pelagic groups which are common to most marine sediments, and which frequently constitute the major portion of these deposits. It is to be expected that the revived interest in the biological studies in this group will soon result in the solution of these and other problems, but it will require much time and diligence on the part of the investigator to eliminate those factors which have caused many to question the value of much of the evidence presented in the past.

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GROUPS HAVING A MAXIMUM NUMBER SET OF INDEPENDENT GENERATORS

BY G. A. MILLER

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS

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A group G is said to have a maximum number set of independent generators whenever it has at least one set of independent generators which involves as many operators as there are prime factors in the order of the group. Each of the operators of such a set is obviously of prime order and if two such operators are of different orders they generate a group whose order is the product of these orders. When two such operators are of the same order they generate a group whose order is either the square of this common order or the product of this common order and a larger prime number which is congruent to unity with respect to this common order. All the operators of the latter group except those in the invariant cyclic subgroup whose order is this larger prime number have the same order.

Every subset composed of k of the operators of a maximum number set of independent generators of such a group generates a subgroup of G whose order is the product of exactly k prime numbers and hence has itself a maximum number set of independent generators. Every operator in such a subset therefore generates a cyclic subgroup which is invariant under all the operators of smaller order contained in this subset. In particular, every operator of highest order in the subset generates an invariant cyclic subgroup under the subset whenever it contains only one operator of this order. When no two of the operators of the subset have the same order the order of the subgroup generated by it is the product of the orders of the operators of the subset but when at least two of these operators have the same order the order of the subgroup generated by it may exceed the product of the orders of these operators.

There is at least one subset composed of operators of smallest order contained in the given maximum number set of independent generators of G which generates a Sylow subgroup of G whose order is a power of the small-

est prime number which divides the order of G since the order of the subgroup generated by a subset cannot be less than the product of the orders of the operators contained in the subset. If a Sylow subgroup of G whose order is a power of the smallest prime number p which divides the order of G is of order p^k and the given maximum number set of independent generators would not contain k operators of order p which generate such a Sylow subgroup then every k of its operators would generate a group whose order would exceed p^k. By adjoining successively an operator from the given maximum number set there would result each time a group whose order is equal to the order of the preceding group multiplied by a larger prime number than p but which divides the order of G. This is obviously impossible since the number of such equal prime numbers could not exceed the index of a power of a prime which is equal to the order of a Sylow subgroup.

From the preceding paragraph it results that a maximum number set of independent generators of G contains k relatively commutative operators which generate a Sylow subgroup of G whose order is a power of the smallest prime number which divides the order of G. It does not necessarily contain any operator which appears in any other Sylow subgroup of G even when the order of G is divisible by various different prime numbers. This is illustrated by the fact that the direct product of an arbitrary number of dihedral groups whose separate orders are the doubles of arbitrary odd prime numbers has a maximum number set of independent generators composed of operators of order 2 while its Sylow subgroup whose order is a power of 2 is of order 2^k, where k is the number of these odd primes, which may be either equal or unequal. It is obvious that in this case a maximum number set of independent generators can also be so selected that each of its operators appears in some Sylow subgroup of G. We proceed to prove that such a selection of the operators of a maximum number set of independent generators is always possible whenever G has a maximum number set of independent generators.

For the sake of simplicity we shall first restrict our attention to the case when the order of G is divisible by only two different prime numbers. When the given maximum number set of its independent generators involves exactly as many operators of lowest order as the index of the highest power of the smaller prime which divides the order of G then it results directly from the preceding proof that G contains two abelian Sylow subgroups such that each is of a type which is of the form 1^k and that each operator in a set of generators of the Sylow subgroup whose order is a power of the larger prime generates a cyclic subgroup which is invariant under G. If the given maximum number set of independent generators of G involves more operators of lowest order than the number of the independent generators of a Sylow subgroup of G whose order is a power of the smallest prime

which divides the order of G we adjoin to the generators of such a Sylow subgroup which appear in the given set a generator of the same order which also appears therein but not in the given Sylow subgroup.

These generators give rise to a group whose order is a prime number times the order of this Sylow subgroup. This group has for a maximum number set of independent generators the given generators of this Sylow subgroup and an operator whose order is this prime number. The cyclic group generated by this operator is invariant under this extended group since it is invariant under the given Sylow subgroup of G. This invariance results from the fact that its two given generators and any other generator of the given Sylow subgroup must generate a group having a maximum number set of three independent generators. If the given maximum number set of independent generators of G involves an operator of the lower order which is not contained in the given extended group we proceed similarly with a second extension. The order of the group resulting from this second extension is equal to the order of the given Sylow subgroup of G multiplied by the square of the largest prime which divides the order of G. This group contains a Sylow subgroup whose order is this square and whose two generators separately generate groups which are invariant under the given Sylow subgroup of G.

The given process can be continued until all the operators of lowest order in the given maximum number set of independent generators of G have been exhausted and as the operators of higher order contained in this set (if any) can be adjoined to those thus obtained so as to obtain an abelian Sylow subgroup whose order is a power of the larger of the prime numbers which divide the order of G. This abelian group is clearly of type which is of the form 1^k and the group generated by each of a set of generating operators is invariant under G. Operators composed of a set of independent generators of each of its two Sylow subgroups clearly constitute a maximum number set of independent generators of G. It should be noted that when operators in the given maximum number set of independent generators are replaced by operators of higher order so as to obtain a new maximum number set of such generators the latter operators generate the same group when adjoined to the given Sylow subgroup as the former.

The given method requires only slight modifications to apply to the general case. After a Sylow subgroup whose order is a power of the smallest prime which divides the order of G has been found the adjoined operators of the maximum number set of independent generators, if it contains such operators, should be so selected as to obtain a group of one of the smallest orders possible. The same is true as regards the additional operators thus selected until we obtain a group which involves a Sylow subgroup whose order is a power of the next to the smallest prime number which divides the order of G. After this is done we proceed similarly with the selection of

additional independent generators of G. This proves that *whenever a group contains at least one maximum number set of independent generators all of its Sylow subgroups are abelian and of type* 1^k *and each of its Sylow subgroups is invariant under all of its Sylow subgroups whose orders are powers of smaller prime numbers.* In particular, it contains only one Sylow subgroup whose order is a power of the largest prime which divides the order of the group.

An interesting special case presents itself when G contains a maximum number set of independent generators which is composed of operators of order 2. In this case the Sylow subgroup whose order is a power of 2 is selected just as the Sylow subgroup whose order is a power of the smallest prime which divides the order of G is selected in the general case. If the group contains additional operators it may be extended so as to obtain a group whose order is divisible by a larger prime number. The operators of the cyclic group generated by an operator whose order is equal to this prime number are transformed either into themselves or into their inverses by all the operators of the given Sylow subgroup and hence they are invariant under a subgroup of index 2 contained in this Sylow group if they are not invariant under this entire group. As an operator of odd prime order cannot be transformed into its inverse by such an operator it results that these operators are relatively commutative under G. Hence the following theorem: *If a group has a maximum number set of independent operators composed of operators of order* 2 *then its operators of odd prime order generate an abelian invariant subgroup.*

From the fact that the independent generators of such a group can be so selected that each of them is of prime order it results that when a group has a maximum number set of independent generators the order of none of its operators is divisible by the square of a prime number. When such a set is composed of operators of order 2 the invariant abelian subgroup generated by its operators of odd prime order can usually be extended in various ways so as to obtain the entire group. The smallest such group results when we adjoin to this subgroup an operator of order 2 which transforms each operator of this subgroup into its inverse. If k is the sum of the number of independent generators in the Sylow subgroups of this abelian subgroup of odd order then the number of the independent generators of G composed of operators of order 2 is $k + 1$ whenever G is this smallest extended group. It is $2k$ when G is the largest possible such extended group based on the given abelian group generated by operators of odd prime order. The objects of the present article are extensions and clarifications of results announced in an article which appeared in the preceding volume of these PROCEEDINGS, pages 333–337.

SPECIAL HOMOLOGY GROUPS

By M. RICHARDSON

INSTITUTE FOR ADVANCED STUDY

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In this note, new invariants of the topological type of a finite group of homeomorphisms are introduced.

1. Let K be a simplicial n-complex and G a finite group of simplicial homeomorphisms $T_0 = 1, T_1, \ldots, T_{m-1}$ of K on itself. By an *invariant subdivision* K' of K is meant a simplicial subdivision such that (a) the T_i permute the simplexes of K' and (b) no vertex v is adjacent to two distinct transforms $T_iv' \neq T_jv'$ of any vertex v' unless $v = T_iv'' = T_jv''$. Such subdivisions $K', K'', \ldots$ exist. Let K denote hereafter a definite invariant subdivision.

Let $a_0, a_1 \ldots, a_{m-1}$ be an arbitrary but fixed set of m integers, not all zero, and define the operator a as follows:[1]

$$a = a_0T_0 + a_1T_1 + \ldots + a_{m-1}T_{m-1}. \qquad (1)$$

A p-chain C_p (over any coefficient group $\mathfrak{G}$) is said to be of *type* a if there exists a p-chain X_p such that $C_p = aX_p$. The p-chains of type a form a subgroup a_p of the group L_p of all p-chains of K. The p-cycles of type a form a subgroup $\mathfrak{A}_p$ of the group Γ_p of all p-cycles of K; in fact $\mathfrak{A}_p = a_pa \frown \Gamma_p$. If C_{p+1} and C_p are both of type a and $FC_{p+1} = C_p$, we write $C_p \overset{a}{\sim} 0$ (read a-homologous to zero). Such relations are called[2] *special homologies* (of type a). The p-chains which are a-homologous to zero form a subgroup $\mathfrak{A}_p^0$ of the group Γ_p^0 of bounding p-cycles; in general, $\mathfrak{A}_p^0$ is a proper subgroup of $\Gamma_p^0 \frown \mathfrak{A}_p$. The group $H_p^a(K, G) = \mathfrak{A}_p - \mathfrak{A}_p^0$ is called the *p-dimensional special homology group of type a* over $\mathfrak{G}$. There is a group $H_p^a(K, G)$ for each choice of the integers $a_0, \ldots, a_{m-1}$. In the special case where $a_0 = 1$ and $a_i = 0$ for $i \neq 0$, $H_p^a(K, G)$ is merely the ordinary homology group $H_p(K)$ over $\mathfrak{G}$. In general, $H_p^a(K, G)$ is not isomorphic to any subgroup of $H_p(K)$. We have, however, the following theorem: if every p-cycle of type a which bounds is a-homologous to zero (that is, if $\mathfrak{A}_p^0 = \Gamma_p^0 \frown \mathfrak{A}_p$), then $H_p^a(K, G)$ is isomorphic to a subgroup of $H_p(K)$.

It is clear that many other types of special homology can be introduced. For example, suppose K^* is the subcomplex of K composed of all points which are fixed under all the T_i. A chain C_p is said to be of type α if there exists a chain X_p such that $C_p = aX_p \bmod K^*$. We can then define special homology groups $H_p^\alpha(K, G)$ of type α in an obvious way.

There are many algebraic relations among the various groups considered which we shall not state here. If we know the automorphisms of the groups L_p induced by the T_i, we can calculate the special homology groups.

2. The special homology groups over any $\mathfrak{G}$ may be considered as the corresponding homology groups of certain abstract complexes associated in an obvious way with K and the transformations T_i. Abstract complex is meant in a sense slightly more general than that of Mayer;[3] that is, a sequence of abelian groups (here, the groups of chains of a particular type) with a boundary homomorphism F such that $FF = 0$. The special homology groups over the integers may be regarded as the corresponding homology groups of abstract complexes, in the sense of Tucker,[4] whose "cells" are the elements of independent bases for the groups of chains of the particular type considered. For example, denote by s the operator ΣT_i; that is, let all $a_i = 1$ in (1). Then the special homology group $H_p^s(K, G)$ of type s over the integers may be considered as the p-dimensional homology group of an abstract complex, in the sense of Tucker,[4] which is isomorphic with the complex k obtained by identifying all points of K which are congruent under the transformations T_i; hence $H_p^s(K, G) \simeq H_p(k)$.

It was proved[5] that the automorphisms induced in the $H_p(K)$ by the T_i suffice to determine the Betti numbers of k over the integers or the integers mod π^u, π a prime not dividing the order[6] m of G. A simple example shows, however, that the automorphisms induced by the T_i in the $H_p(K)$ and in the fundamental group of K together do not determine the torsion group of k, and hence do not determine completely the group $H_p^s(K, G)$.

3. Clearly, continuity considerations have not entered up to this point; hence everything said so far would apply equally well if K were merely an abstract simplicial complex, say. The following theorem gives topological significance to the special homology groups.

THEOREM. *The special homology groups are invariants of the topological type of* G.

This means that if K^0 admits a group G^0 of simplicial homeomorphisms T_i^0 and if there exists a homeomorphism τ mapping K on K^0 such that $T_i^0 = \tau T_i \tau^{-1}$ for all i, then the special homology groups $H_p^t(K, G)$ and $H_f^t(K^0, G^0)$ are isomorphic, for any type t.

Various known proofs of the invariance of the homology groups can be so modified as to yield proofs of this theorem. In particular, the special homology groups can be invariantly characterized either by using singular chains of type t or by using a sequence of "invariant" coverings (that is, coverings whose nerves are invariant in the sense of §1) composed of the stars of the vertices of successive invariant subdivisions with meshes converging to zero. Invariant coverings in general can also be used to apply these ideas to abstract spaces by means of the homology theory of Čech. In this connection, cf. a forthcoming paper of P. A. Smith in the *Annals of Math.*

[1] All expressions of the form (1) where the a_i range independently over all integers constitute the group-ring of G which has been used in recent work of K. Reidemeister, W. Franz and G. de Rham (cf. de Rham, *Matem. Sbornik, N. S. I*, 737–742 (1936) for references). Their work, however, appears to have little or nothing in common with this note.

[2] The notion of special homology was introduced for the special case where G is of order 2 and $\mathfrak{H}$ is the group of integers mod 2 by P. A. Smith, *Proc. Nat. Acad. Sci.*, **19**, 612–618 (1933).

[3] W. Mayer, *Monatshefte Math. Physik*, **36**, 1–42 (1929). The more general definition used here will be exploited in detail by Mayer in a forthcoming set of notes.

[4] A. W. Tucker, *Ann. Math.*, **34**, 191–243 (1933).

[5] M. Richardson, *Duke Math. Jour.*, **1**, 50–69 (1935); Correction, *Ibid.*, **3**, 382 (1937).

[6] For the case where G is generated by a transformation of period m, the homology groups over the integers mod m of k, of K^*, and of k mod the subset k^* of k corresponding to K^* will be discussed in a joint paper by P. A. Smith and the author to appear elsewhere. Certain special homology groups play an important rôle in that paper. The case $m = 2$ was treated by P. A. Smith, loc. cit., and by M. Richardson, *Bull. Amer. Math. Soc.*, **41**, 528–534 (1935), under certain restrictive hypotheses.

LOG π AND OTHER BASIC CONSTANTS

BY HORACE S. UHLER

SLOANE PHYSICS LABORATORY, YALE UNIVERSITY

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Consideration of the possibility of extending the 70-figure part of my recent table of reciprocals of factorials[1] for values of n greater than 369 or of checking discrete higher data in this table indicated that Stirling's asymptotic approximation formula[2] for $\log \Gamma(n)$ was best adapted to the calculation of isolated values of $n!$ to a large number of significant figures. Thanks to the labors of J. C. Adams,[3] S. Z. Serebrennikov[4] and D. H. Lehmer[5] the Bernoulli numbers occurring in this formula are known completely as far as n equal to 110. On the other hand, the term $\frac{1}{2} \log 2\pi$ in Stirling's series has set in the past a fairly low limit to the number of figures obtainable because $\log_e \pi$ and $\log_{10} \pi$ are given correctly in the literature to only 48 and 60 decimal places, respectively. In order to raise this limit to a level at which it will probably not constitute for some years to come a hindrance to those whose researches depend upon the existence of very extended numerical data, I decided to calculate the values of $\log_e \pi$ and $\log_{10} \pi$ to nearly 215 decimal places. Since even a single false digit in the published value of a basic constant can cause incalculable loss of time and energy, and since unfortunately the literature of large numbers abounds in erroneous figures, it seems desirable not only to present

briefly the details of the method followed in the present arithmetical investigation but also to emphasize the care and precautions which were taken in the earnest endeavor to obtain reliable results.

The formula underlying the work is

$$\pi = (355/113)(1 - .0003/3533)f \qquad (1)$$

It was derived from the approximation expression for π, discovered by Ramanujan,[6] by the introduction of the correction factor f which makes formula (1) exact. A rough value for $1 - f$ is 347×10^{-18}. Hence the actual computation of $\log_e \pi$ reduced to that of the Napierian logarithms of 71, 113, $1 - .0003/3533$ and f. The evaluation of the first two of these logarithms was arranged to depend primarily upon the logarithms of 2, 3, 5 and 7, since the latter four have been finally computed by Adams[7] to about 273 places of decimals.

The series

$$\log \frac{p}{q} = 2\left\{\frac{p-q}{p+q} + \frac{1}{3}\left(\frac{p-q}{p+q}\right)^3 + \frac{1}{5}\left(\frac{p-q}{p+q}\right)^5 + \ldots\right\} \equiv 2S[(p-q)/(p+q)] \qquad (2)$$

is very convenient when $p - q = 1$, a condition which is fulfilled by the pairs of numbers (5041, 5040) and (226, 225). Since $5041 = 71^2$ and $5040 = 2^4 3^2 5 \cdot 7$, the logarithm of 71 was calculated from the equation

$$\log_e 71 = 2\log_e 2 + \log_e 3 + (\log_e 5 + \log_e 7)/2 + S(1/10081),$$

in which the last term denotes the series between the braces of formula (2) when $p = 5041$ and $q = 5040$. Similarly, letting $p = 226 = 2 \cdot 113$ and $q = 225 = 3^2 5^2$, the logarithm of 113 was obtained from the relation

$$\log_e 113 = 2(\log_e 3 + \log_e 5) - \log_e 2 + 2S(1/451).$$

For the factors f (used as $1 - \epsilon$) and $1 - .0003/3533$ it was necessary to employ the series

$$-\log(1 - x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots \qquad (3)$$

The noteworthy advantage of Ramanujan's expression is that when the fractions $3/(3533 \times 10^4)$ and ϵ are substituted for x in formula (3) the series converges respectively with satisfactory speed and with great rapidity. Incidentally since by its defining fraction,

$$(113)(3533 \times 10^4)\pi/(355)(35329997),$$

f is directly proportional to π, ϵ is also an interminable decimal fraction which must be carried to a greater number of places than the arbitrary limit (215) set for $\log_e \pi$. Hence the integral powers of ϵ had to be ob-

tained by successive multiplication instead of repeated division as was the case for the series the denominators of whose terms were integral multiples of powers of the finite numbers 451, 3533 and 10081.

It is now appropriate to consider the checks which I have applied both to the data computed by Adams and to the new numbers involved in the present work. Although the probability of finding an error in Adams' celebrated results may be considered as extremely small, nevertheless a careful study of his two original papers on this subject caused me to doubt if he had subjected his final values of $\log_e 3$, $\log_e 5$ and $\log_e 7$ to any direct check. In the equation of condition $a - 2b + c = d + 2e$, which Adams found was satisfied to 274 places of decimals, the letters do not symbolize the logarithms of 2, 3, 5 and 7 but have the following meanings: $a = \log_e(10/9)$, $b = \log_e(25/24)$, $c = \log_e(81/80)$, $d = \log_e(50/49)$ and $e = \log_e(126/125)$. It was possible to make a mistake in solving such linear equations as $\log_e 3 = 11a - 3b + 5c$, $\log_e 7 = 19a - 4b + 8c + e$, etc. Adams' work on Euler's constant, which included the independent computation of $\log_e 500$ and $\log_e 1000$, placed the first 263 figures of $\log_e 2$ in quite a different category. Finally, since I failed to find recorded in the literature any case in which Adams' published data had been used or corroborated, the ever present danger of a typographical error in the higher decimals (above 100) still existed.

Since $4375 = 5^4 \cdot 7$, and $4374 = 2 \cdot 3^7$, formula (2) requires that the condition

$$(4 \log_e 5 + \log_e 7 - \log_e 2 - 7 \log_e 3)/2 - S(1/8749) = 0$$

be fulfilled. The terms of $S(1/8749)$ were calculated to 233 decimal places. Let S_∞ denote the limit $a/(1 - r)$ of a simple geometrical series. In the present case it was found that $S_\infty - S_{30} = 8749/76545000 - S_{30} = 3 \times 10^{-233}$, where S_{30} means the sum of the 30 requisite terms of the series. When Adams' logarithms of 2, 3, 5 and 7, and my value of $S(1/8749)$ were substituted in the equation of condition given above, the left-hand member equaled $1 \cdot 10^{-233}$. A very simple check on all the figures of $\log_e 2$ and $\log_e 3$ was available due to the fact that in August of the year 1900 I finished a longhand computation of π which agreed perfectly with the accepted datum. This earlier work afforded unimpeachable values for the sums of the positive terms and for the sums of the negative terms of both $\tan^{-1}(1/5)$ and $\tan^{-1}(1/239)$. Let $s(+)$ and $s(-)$ denote the arithmetical values of the two sums in the case of $\tan^{-1}[(p - q)/(p + q)]$, then, in the notation introduced above in formula (2),

$$\log(p/q) = 2S[(p - q)/(p + q)] = 2[s(+) + s(-)].$$

For $p = 3$ and $q = 2$ it was found that $\log_e 3 - \log_e 2 - 2S(1/5) = 2.5 \times 10^{-275}$. Since my value of $\tan^{-1}(1/5)$ agreed with that given by Wm.

Rutherford[8] to 287 places of decimals, the values of $\log_e 2$ and $\log_e 3$ are alone responsible for the resultant error 2.5×10^{-275} which is quite as small as, if not smaller than, Adams' own estimate.

Attention will now be turned to the tests to which the new arithmetical data have been subjected. Throughout the entire work every quotient obtained by machine division was checked at least once by machine multiplication, and the results of mental short-division were gone over on two or three occasions which were far apart in time. The separate terms of the series for $\log_e 71$ were carried to 215 decimal places, and it was found that $S_\infty - S_{27} = 10081/101626560 - S_{27} = 3.2 \times 10^{-216}$. The same approximation was used in the case of $\log_e 113$, and the geometrical progression check yielded $S_\infty - S_{40} = 451/203400 - S_{40} = 3.8 \times 10^{-215}$. This difference is not at all excessive because 40 terms contributed to the figures in the highest decimal places, and certain of the digits occupying the 215th place were increased by unity conformably to the usual convention. The computation of $\log_e(1 - .0003/3533)$ was terminated at the 215th place of decimals, and $S_\infty - S_{30} = 3/35329997 - S_{30} = 0 \times 10^{-215}$. The series for $\log_e(1 - \epsilon)$ converged so rapidly that only 14 terms were required in carrying the calculation to about 222 decimal places. This higher degree of approximation was attained advisedly because experience has taught me to segregate the digits of a very long number into groups of 8 when multiplying with the 10-column Monroe machine, and it was necessary to employ 28 such groups in order to avoid uncertainty about the digits in the 214th and 215th decimal places. The check relation in this case is that $f \cdot (1 + \sum_1^{14} \epsilon^n)$ should equal unity. The value found was $1 + 9 \times 10^{-224}$. Since $\log_e 5$ is known to a still higher degree of approximation, it now follows from formula (1) that the limit of accuracy of my value of $\log_e \pi$ depends only upon the reliability of the final digits of the three numbers $\log_e 71$, $\log_e 113$ and $\log_e(1 - .0003/3533)$.

In September, 1937, I tested anew all phases of the work which were not guaranteed by the different values of S_∞, more specifically the decimal positions of the leading figures of all the terms of the several secondary series were verified, the terms of these series were multiplied by their respective small integers and the products compared with the corresponding terms of the primary geometric series, etc. No error of any kind was detected.

The last stage of the investigation consisted in transforming from the Napierian system to the base 10 not only my new data but also, for sake of completeness, about 230 figures of the logarithms of 2, 3, 5 and 7, originally computed by Adams and given heretofore only to the base e. This procedure not only enhanced the practical value of the data but it also afforded an additional check on the greater part of the work as a whole.

The act of machine multiplying each of the natural logarithms by the modulus was performed twice. Also the value found for $\log_{10} \pi$ was multiplied by $\log_e 10$, that is, by the reciprocal of the modulus. The number obtained agreed perfectly with $\log_e \pi$, thus showing that the earlier multiplication had been performed correctly, and also that $\log_e 10$ is free from error to the number of decimal figures employed. Next, $\log_{10} \pi$ was computed from the equation

$$\log_{10} \pi = 3 \log_{10} 2 - \log_{10} 3 + (\log_{10} 7 - \log_{10} 5)/2 - MN,$$

where M and N denote respectively the modulus and the single number resulting from the expression

$$2S(1/451) - S(1/10081) - \log_e (1 - \epsilon) - \log_e (1 - .0003/3533).$$

The two calculations of $\log_{10} \pi$ led to identical values inclusive of the 215th place of decimals. It was also found that $\log_{10} 2 + \log_{10} 5 = 1 - 1.5 \times 10^{-232}$.

In conclusion I desire to thank my mathematical colleague, John W. Wrench, Jr., for having called my attention both to the existence of Ramanujan's approximation expression for π and to the obvious relation between the terms of the series for $\tan^{-1} x$ and $\frac{1}{2} \log (1 + x)/(1 - x)$. He has recently computed, without the aid of a machine, the reciprocal of π to about 254 decimal places. It was checked by me, by performing with the machine the multiplication of his result by π. The product found was $1 + 2.4 \times 10^{-254}$. By courtesy this latest value of $1/\pi$ is incorporated in the table given below. The most extended earlier approximation for $1/\pi$ was calculated to 140 decimal places by G. Paucker.[9] His value is less than the newest by 298×10^{-140}.

TABLE 1

$\frac{1}{2} \log_{10} 2\pi =$

0.39908 99341 79057 52478 25035 91507 69595 02099 34102 92127
54937 22240 48052 60276 50681 98859 20712 37319 29123 86972
05056 53359 18953 19647 11174 66092 97915 99471 78837 75112
84289 65464 24863 18097 51691 73867 72077 27707 09582 41828
29137 91498 4112(207)

$\log_e \pi =$

1.14472 98858 49400 17414 34273 51353 05871 16472 94812 91531
15715 13623 07147 21377 69884 82607 97836 23270 27548 97077
02009 81222 86979 89159 04820 55279 23456 58727 90810 78810
28682 52763 93914 26634 59029 02484 77335 88699 37789 20311
96308 24756 794(0 5)

$\log_{10} \pi =$

0.49714 98726 94133 85435 12682 88290 89887 36516 78324 38044
24461 34053 49992 49471 12089 55267 46555 47386 46429 12223
69428 58999 23596 43915 12872 53374 62308 48343 60752 16520
99021 80283 46762 10775 69356 85915 70723 39384 75663 65266
29294 04414 2052(870)

$\log_{10} 2 =$

0.30102 99956 63981 19521 37388 94724 49302 67681 89881 46210
85413 10427 46112 71081 89274 42450 94869 27252 11818 61720
40684 47719 14309 95379 09476 78811 33523 50599 96923 33704
69557 50645 02964 25419 34026 61819 73431 16029 43501 18390
28981 78582 61715 44395 31861 92904 (62)

$\log_{10} 3 =$

0.47712 12547 19662 43729 50279 03255 11530 92001 28864 19069
58648 29865 64030 52291 52783 66112 30429 68355 64761 63015
10464 69276 82520 45893 56296 91422 25227 35129 03437 06071
52940 99332 42951 85317 13574 89785 20429 54767 99178 65255
70688 28821 46923 23368 60689 63825 (7)

$\log_{10} 5 =$

0.69897 00043 36018 80478 62611 05275 50697 32318 10118 53789
14586 89572 53887 28918 10725 57549 05130 72747 88181 38279
59315 52280 85690 04620 90523 21188 66476 49400 03076 66295
30442 49354 97035 74580 65973 38180 26568 83970 56498 81609
71018 21417 38284 55604 68138 07095 (37)

$\log_{10} 7 =$

0.84509 80400 14256 83071 22162 58592 63619 34835 72396 32396
54065 03634 95371 82534 39902 07916 60661 11527 84748 85733
41424 31007 53543 45586 24160 61706 83618 92508 57975 83788
44207 69978 03075 65406 40944 61350 30544 25266 34969 30535
81510 61615 97219 40630 33871 84801 (1)

$\log_{e} 17 =$

2.83321 33440 56216 08024 95346 17873 12653 55882 03012 58574
47872 97237 73788 22925 75800 93128 09120 94868 03750 29475
18348 26204 71870 57291 39759 28419 46738 36429 97545 65742
02127 12599 13208 07209 04790 76471 68172 51666 60296 60850
69091 96813 96134 51492 95164 19209 44718 69393 25481 33184
68944 45037 58003 15646 0300(9)

$\log_{10} 17 =$

1.23044 89213 78273 92854 01698 94328 33703 00075 67378 42504
63973 80368 48234 46940 62257 11818 57956 84670 09846 51395
05485 78524 97989 65012 17473 37431 15114 67405 38882 80051
55012 35617 61522 77017 71733 74305 45360 56161 02806 91916
25780 06320 65897 46206 33195 11359 (4)

$\log_e 71 =$

4.26267 98770 41315 42132 94545 32513 03409 67595 76526 71056
61081 21425 80202 73515 06824 23036 59662 43324 27263 51335
40946 35034 57211 12385 45755 79963 96882 06266 76523 92546
74923 81006 29748 92487 16955 91222 00442 26858 02841 10267
16204 52238 035(8 7)

$\log_{10} 71 =$

1.85125 83487 19075 28609 28294 35035 42913 52704 19901 60039
19762 76649 87113 13708 89795 68141 74999 31707 14353 56346
75473 76927 (6)

$\log_e 113 =$

4.72738 78187 12340 56858 21314 93616 02167 20293 30388 70299
37947 40348 03534 35739 05559 05007 08361 90102 49987 20779
62663 12060 94546 99147 97069 13900 69288 41850 81695 35298
33887 03009 48395 69536 80408 87907 81456 08500 82081 61919
52260 86331 124(2 9)

$\log_{10} 113 =$

2.05307 84434 83419 72279 52270 28609 44818 47783 83623 62209
73395 15705 38185 81852 95591 02402 68378 23275 09682 93555
33103 21934 (8)

$1/\pi =$

0.31830 98861 83790 67153 77675 26745 02872 40689 19291 48091
28974 95334 68811 77935 95268 45307 01802 27605 53250 61719
12145 68545 35159 16073 78582 36922 29157 30575 59348 21463
39967 84584 79933 87481 81551 46155 49279 38506 15377 43478
57924 34795 32338 67247 80483 44725 80236 64760 22844 53995
114(4)

$\pi^2 =$

9.86960 44010 89358 61883 44909 99876 15113 53136 99407 24079
06264 13349 37622 00448 22419 20524 30017 73403 71855 22318
24025 91377 40231 44077 77234 81220 30046 72761 06176 77985
19766 09903 99856 20657 56305 71506 04123 28403 28780 86935
27693 42164 93966 65715 19044 53873 52617 79413 82025 82605
81693 41251 5(5)

This table also contains a new value of $\log_e 17$ which was derived from my old values of $s(+)$ and $s(-)$ for $\tan^{-1}(1/239)$ and which is certainly correct to 273 places of decimals, the limit being set solely by the basic data of Adams. The Napierian logarithms of 71 and 113 were converted to the base 10 only to about 110 decimal places because the transformed numbers are not particularly fundamental. It is interesting to find that H. M. Parkhurst[10] gives 04 and 90 instead of 05 and 75 for the digits in places 101 and 102, respectively, for $\log_{10} 17$ and $\log_{10} 71$.

The last constant included in the table, π^2, was computed by me to about 262 decimal places in order to test and extend the number given by Serebrennikov[11] to 220 places. The first 218 significant figures of the two results were identical; his value exceeded mine by 595×10^{-220}. Because of this agreement the machine multiplication was repeated only from the twenty-seventh "sum of products of pairs of eight-figure groups" to the thirty-third sum.

For the numbers tabulated above, the terminal figures whose accuracy may be at all debatable are enclosed in parentheses.

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[2] H. T. Davis, *Tables of the Higher Mathematical Functions*, **1**, 180–183 (1933)

[3] *Scientific Papers of John Couch Adams*, **1**, 426–458 (1896).

[4] S. Z. Serebrennikov, *Mém. Acad. Imp. Sci. St.-Pétersbourg*, **16**, No. 10 (1905); *Ibid.*, **19**, No. 4 (1906).

[5] D. H. Lehmer, *Duke Math. Jour.*, **2**, 460–464 (1936).

[6] *Collected Papers of Srinivasa Ramanujan*, P. 35 (1927).

[7] J. C. Adams, *Proc. Roy. Soc. London*, **42**, 22–25 (1887).

[8] Wm. Rutherford, *Proc. Roy. Soc.*, **6**, 274, 275 (1853).

[9] G. Paucker, Grunert's *Archiv Math. Physik*, **1** (1), 9–11 (1841).

[10] H. M. Parkhurst, *Astronomical Tables Comprising Logarithms from 3 to 100 Decimal Places*, New York (1876).

[11] See the second reference under 4.

EINSTEIN SPACES IN A SPACE OF CONSTANT CURVATURE

BY AARON FIALKOW[1]

DEPARTMENT OF MATHEMATICS, BROOKLYN COLLEGE

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In this note, we announce certain new results which give a classification of the real Einstein spaces[2] E_n which can be imbedded in a real space of constant curvature S_{n+1}. The proofs of these theorems will be given elsewhere. We do not restrict ourselves to the case where the first fundamental forms of E_n and S_{n+1} are positive definite. We assume that the imbedding is given by the equations

$$y^\alpha = y^\alpha(x^i) \quad \begin{matrix} i = 1, 2, \ldots, n. \\ \alpha = 1, 2, \ldots, n+1. \end{matrix}$$

where the $y^\alpha(x^i)$ are real functions of class C^3 and the rank of $\left\| \frac{\partial y^\alpha}{\partial x^i} \right\|$ is n. In some cases, we also assume that the principal normal curvatures are real and that none of the lines of curvature of E_n is tangent to a null vector.[3] If these last two conditions are fulfilled we call E_n a *proper*

hypersurface of S_{n+1} and call the imbedding a *proper* imbedding. If the first fundamental form of the enveloping S_{n+1} is definite neither of the two exceptional cases mentioned above can occur and therefore every hypersurface is proper. Thus we find *all* the Einstein spaces which are hypersurfaces of a space of constant curvature whose first fundamental form is definite (in particular, of a Euclidean space). These results generalize theorems obtained by previous writers to whom references are given below.

An Einstein space E_n is defined as a Riemann space V_n whose mean curvature λ is a constant at each point. If $n > 2$, λ is a constant throughout the space. The case $n = 2$ presents no problem since every V_2 is also an E_2. Neither do we consider the case $n = 3$. For every E_3 is an S_3 and the possibility and manner of its imbedding in a given S_4 is well known. We therefore suppose $n \geqq 4$. Our main result is given by

THEOREM 1. *Let* $\mathrm{E_n}$ $(\mathrm{n} \geqq 4)$ *be a real Einstein space of mean curvature* λ *which is a proper*[4] *hypersurface of a real space* $\mathrm{S_{n+1}}$ *of constant curvature* $\mathrm{K_0}$. *Let the signatures of the first fundamental forms of* $\mathrm{E_n}$ *and* $\mathrm{S_{n+1}}$ *be* s *and* $\mathrm{s + e}$, *respectively.*[5] *If* $\mathrm{(a)\ e\lambda > e(n-1)K_0}$, *then* $\mathrm{E_n}$ *is an* $\mathrm{S_n}$ *of Riemann curvature* K *where* $\mathrm{eK > eK_0}$. *Conversely, each* $\mathrm{E_n}$ *of type* (*a*) *can be imbedded as a hypersurface of* $\mathrm{S_{n+1}}$ *and has indeterminate lines of curvature and constant normal curvature.* (*b*) $\mathrm{\lambda = (n-1)K_0}$, *then* $\mathrm{E_n}$ *is an* $\mathrm{S_n}$ *of Riemann curvature* $\mathrm{K_0}$. *Each* $\mathrm{E_n}$ *of type* (*b*) *can be imbedded either as a totally geodesic or a developable hypersurface of* $\mathrm{S_{n+1}}$. (*c*) $\mathrm{e\lambda < e(n-1)K_0}$, *then* $\mathrm{eK_0 > 0}$ *and* $\mathrm{\lambda = (n-2)K_0}$. $\mathrm{E_n}$ *contains* $\mathrm{\infty^{n-p}S_p}$ *of Riemann curvature* $\dfrac{\mathrm{n-2}}{\mathrm{p-1}}\,\mathrm{K_0}$ *and also, orthogonal to the* $\mathrm{S_p}$, $\mathrm{\infty^{p}S_{n-p}}$ *of Riemann curvature* $\dfrac{\mathrm{n-2}}{\mathrm{n-p-1}}\,\mathrm{K_0}$ (p *is any fixed integer so that* $2 \leqq p \leqq \mathrm{n-2}$). *The* $\mathrm{S_p}$ *and* $\mathrm{S_{n-p}}$ *are totally geodesic in* $\mathrm{E_n}$. *Each* $\mathrm{E_n}$ *of type* (*c*) *can be imbedded in* $\mathrm{S_{n+1}}$. *The curves in any* $\mathrm{S_p}$ *or any* $\mathrm{S_{n-p}}$ *are the lines of curvature of* $\mathrm{E_n}$. *The principal normal curvature of* $\mathrm{E_n}$ *has one constant value on all the* $\mathrm{S_p}$ *and another on all the* $\mathrm{S_{n-p}}$.

The result stated in (*a*) was recently proved by Thomas[6] and Cartan[7] subject to the restriction that S_{n+1} is a Euclidean space. The theorem in (*b*) was proved by Kasner[8] for a Euclidean S_{n+1} with $n = 4$ and for any dimensionality by Schouten and Struik.[9] Eisenhart[10] showed that the result is true in any flat space. The existence of Einstein spaces of type (*c*) is proved here for the first time. The possibility of an E_4 of type (*c*) in a Euclidean S_5 was first indicated by Kasner.[11] He asked the question whether such E_4 actually existed. We answer this question in the negative in

COROLLARY I. *There are no Einstein spaces of dimensionality* $\mathrm{n} \geqq 4$ *of negative mean curvature which are hypersurfaces of a Euclidean space.*

There is an analogous theorem for spaces of constant negative curvature whose first fundamental form is positive definite. Another immediate consequence of Theorem 1 is

COROLLARY II. *Every Einstein space of dimensionality* $n \geqq 4$ *which is a proper*[4] *hypersurface of a flat space is a hypersphere, hyperplane or developable hypersurface.*

The first fundamental form and the equations of the imbedding of an E_n of types (*a*) or (*b*) in an S_{n+1} are known. We now state the corresponding results for type (*c*).

COROLLARY III. *The first fundamental form of an* E_n *of type* (*c*) *may be written as*

$$ds^2 = \frac{e_1(dx^1)^2 + \ldots + e_p(dx^p)^2}{\left[1 + \frac{K_1}{4}(e_1x^{1^2} + \ldots + e_px^{p^2})\right]^2} + \frac{e_{p+1}(dx^{p+1})^2 + \ldots + e_n(dx^n)^2}{\left[1 + \frac{K_2}{4}(e_{p+1}x^{p+1^2} + \ldots + e_nx^{n^2})\right]^2}$$

where $K_1 = \frac{n-2}{p-1}K_0$, $K_2 = \frac{n-2}{n-p-1}K_0$ *and each e is* $+1$ *or* -1.

THEOREM 2. *An* E_n *of type* (*c*) *in an* S_{n+1} *is indeformable and is imbedded in the* S_{n+1} *by means of the algebraic equations*

$$e_1z^{1^2} + \ldots + e_{p+1}z^{p+1^2} = \frac{p-1}{n-2} \cdot \frac{1}{K_0}$$

$$e_{p+2}z^{p+2^2} + \ldots + e_{n+2}z^{n+2^2} = \frac{n-p-1}{n-2} \cdot \frac{1}{K_0}$$

where the e's are each $+1$ *or* -1 *and the* S_{n+1} *is defined by*

$$e_1z^{1^2} + \ldots + e_{n+2}z^{n+2^2} = \frac{1}{K_0}.$$

Thus an E_n of type (*c*) is also the intersection of two spherical hypercylinders in a flat space of $(n + 2)$ dimensions. For $n = 4$, this situation was discussed by Kasner[12] who showed that the E_4 whose first fundamental form is

$$\frac{dx^{1^2} + dx^{2^2}}{\left[1 + \frac{K}{4}(x^{1^2} + x^{2^2})\right]^2} + \frac{dx^{3^2} + dx^{4^2}}{\left[1 + \frac{K}{4}(x^{3^2} + x^{4^2})\right]^2}$$

may be imbedded in a Euclidean S_6 by means of equations analogous to those of Theorem 2.

He also found that this space is the only E_4 whose first fundamental form is the sum of two forms, one involving only p of the variables, the other involving the remaining $4-p$ variables. When the first fundamental form of a V_n may be written as

$$ds^2 = g_{\alpha\beta}(x^\alpha)dx^\alpha dx^\beta + g_{\gamma\delta}(x^\gamma)dx^\gamma dx^\delta \quad \begin{matrix} \alpha, \beta = 1, 2, \ldots, p \\ \gamma, \delta = p+1, p+2, \ldots, n, \end{matrix}$$

the form is called *separable* and the two forms $g_{\alpha\beta}dx^\alpha dx\gamma^\beta$ and $g_{\gamma\delta}dx^\alpha dx^\delta$ are called its *components*. It is known that the subspaces x^α = constant as well as the subspaces x^γ = constant are totally geodesic in V_n. It follows from Corollary III that the first fundamental form of an E_n of type (c) is separable and each component represents a space of constant curvature. However, if $n > 4$, other E_n's than those of type (c) exist whose first fundamental form is separable. This follows from

THEOREM 3. *If the first fundamental form of an Einstein space of mean curvature* λ *is separable, each component is the first fundamental form of an Einstein space of mean curvature* λ. *If one component involves only one differential,* $\lambda = 0$. *Conversely, only the first fundamental forms of Einstein spaces are separable in this manner.*

By a repeated application of this theorem, we may obtain an obvious generalization which applies to an E_n whose first fundamental form is separable into more than two components.

Closely related to the problem of finding the actual imbedding of an E_n in an S_{n+1} as considered in the preceding paragraphs is the algebraic characterization of such spaces. In a recent paper, Allendoerfer[13] gave such a characterization of the E_n's with $\lambda \not\equiv 0$ which may be imbedded in a flat space. We easily show by his methods that it is possible to obtain a similar characterization of the E_n's in an S_{n+1} of Riemann curvature K_0 if $\lambda \not\equiv (n-1)K_0$. However, it is open to question whether this algebraic characterization applies to any E_n's which are not discussed above; i.e., those E_n's for which the imbedding is not proper. Allendoerfer's proof depends upon a theorem due to Thomas.[14] This theorem states that under certain general conditions, the Codazzi equations for a hypersurface of a flat space are consequences of the Gauss equations. It is easy to show that Thomas' theorem is true even when the enveloping space is not a flat space but any Riemann space.

[1] Most of the results of this paper were obtained while the author was a National Research Fellow at Princeton University and the Institute for Advanced Study.

[2] We denote an n-dimensional Riemann space, Einstein space and space of constant curvature by V_n, E_n and S_n, respectively.

[3] Brinkmann has shown that Einstein spaces E_4 exist whose first fundamental form is indefinite which may be imbedded in a flat space S_5 so that all of the lines of curvature are tangent to null vectors. (Cf. H. W. Brinkmann, "Conformal Mapping of Einstein Spaces," *Math. Ann.*, **94**, 140–141 (1925).

[4] The assumption "proper" is not necessary if the first fundamental form is definite.

[5] Similarly e may be defined for any V_n in a V_{n+1} and is always $+1$ or -1. The value of e depends upon the character of the unit normal to V_n in V_{n+1}. If the first fundamental form of V_{n+1} is positive definite, $s = n$ and $e = +1$.

[6] T. Y. Thomas, "On closed spaces of constant mean curvature," *Amer. Jour. Math.*, **58**, 702–704 (1936).

[7] T. Y. Thomas, "Extract from a Letter by E. Cartan Concerning My Note: On closed spaces of constant mean curvature," *Amer. Jour. Math.*, **59**, 793–794 (1937).

[8] E. Kasner, "The Impossibility of Einstein Fields Immersed in a Flat Space of Five Dimensions," *Amer. Jour. Math.*, **43**, 126 (1921).

[9] J. A. Schouten and D. J. Struik, "On Some Properties of General Manifolds Relating to Einstein's Theory of Gravitation," *Amer. Jour. Math.*, **43**, 215 (1921).

[10] L. P. Eisenhart, *Riemannian Geometry*, 199–200 (1926).

[11] E. Kasner, "Geometrical theorems on Einstein's cosmological equations," *Amer. Jour. Math.*, **43**, 219 (1921).

[12] E. Kasner, "An Algebraic Solution of the Einstein Equations," *Trans. Amer. Math. Soc.*, **27**, 103–104 (1925); see *Proc. Nat. Acad. Sci.*, **11**, 95 (1925).

[13] C. B. Allendoerfer, "Einstein Spaces of Class One," *Bull. Amer. Math. Soc.*, **43**, 265–270 (1937).

[14] T. Y. Thomas, "Riemann Spaces of Class One and Their Characterization," *Acta Math.*, **67**, 191 (1936).

CLASSIFICATION OF ELEMENT TRANSFORMATIONS BY MEANS OF ISOGONAL AND EQUI-TANGENTIAL SERIES

BY EDWARD KASNER AND JOHN DE CICCO

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY AND BROOKLYN COLLEGE

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Introduction.—This paper is a continuation of a paper by the senior writer entitled "The geometry of isogonal and equi-tangential series," published in the Transactions of the American Mathematical Society, vol. 42, no. 1, pp. 94–106. This joint paper may be read independently of the preceding paper.

First, we define two simple operations or transformations on the lineal elements of the plane. A *turn* T_α converts each element into one having the same point and a direction making a fixed angle α with the original direction. By a *slide* S_k, the line of the element remains the same and the point moves along the line a fixed distance k.

We term ∞^1 elements to be a *series* of elements; this includes a union (curve or point) as a special case. By applying a turn T_α to the elements of a union, we obtain a series which we call an *isogonal* series. When a slide S_k is applied to the elements of a union, the resulting series is said to be an *equi-tangential* series.

In the earlier paper, it was shown that the totality of all element transformations may be divided into two distinct classes with respect to the number of isogonal (or equi-tangential) series which are carried into isogonal (or equi-tangential) series; namely, 1°—those which carry all ∞^∞ isogonal (or equi-tangential) series into isogonal (or equi-tangential) series. Any transformation of this class is the product of a conformal transformation by a turn (or the product of an equi-long transformation by a magnification by a slide); 2°—those which carry no more than ∞^4 isogonal (or equi-tangential) series into isogonal (or equi-tangential) series. In this paper, it is our purpose to give a more extensive classification of the element transformations with respect to the precise number of isogonal (or equi-tangential) series which are transformed into isogonal (or equi-tangential) series.

UNIONS INTO ISOGONAL SERIES.—THEOREM 1. *All element transformations are classified into four distinct classes with respect to the number of unions which are converted into isogonal series.*

The classes are described as follows:

1°. Those transformations which convert all (∞^∞) unions into isogonal series. Any transformation T of this class is the product of a contact transformation by a turn. These transformations form a group.

2°. Those which convert a single two-parameter (∞^2) family of unions into isogonal series, and no other family of unions of higher or equal number of parameters. Any transformation T of this class is characterized by the following three properties: (a) All points are converted into points; (b) If C is the associated contact transformation (formed by extending the point transformation), the angle θ between the elements $T(E)$ and $C(E)$ is the same for every element E on any point P; (c) The angle θ is not the same for all points P.

The single two-parameter family of unions which become isogonal series are, of course, the ∞^2 points which become points. Also, for any transformation of this class, there exists a single one-parameter (∞^1) family of unions which become isogonal series which are not all unions.

3°. Those which convert two two-parameter ($2\infty^2$) families of unions into isogonal series, and no other families of unions of higher or equal number of parameters. Any transformation T of this class is such that it carries ∞^2 unions into points, and such that it does not belong to the classes 1° or 2°.

One of the two two-parameter families of unions which become isogonal series is the set of ∞^2 unions which become points. The other two-parameter family of unions become isogonal series (not all unions).

4°. Those which convert one three-parameter (∞^3) family of unions into isogonal series. Any transformation T of this class is characterized by the fact that there do not exist ∞^2 unions which are converted into points. This is the most general case.

ISOGONAL SERIES INTO ISOGONAL SERIES.—THEOREM 2. *All element transformations are classified into four distinct classes with respect to the number of isogonal series which are transformed into isogonal series.*

These classes are characterized as follows:

1°. Those which convert all (∞^∞) isogonal series into isogonal series. Any transformation T of this group is the product of a conformal transformation by a turn.

2°. Those which convert two two-parameter ($2\infty^2$) families of isogonal series into isogonal series, and no other family of isogonal series of higher or equal number of parameters. Any transformation T of this class is characterized by the following four properties: (a) All points are transformed into points; (b) The associated contact transformation C of T is conformal; (c) The angle θ between the directions of $C(E)$ and $T(E)$ is the same for all elements E on any point P; (d) The angle θ is not the same for all points P.

One of the two two-parameter families of isogonal series which become isogonal series is the set of ∞^2 points which are transformed into points. The other two-parameter family of isogonal series become isogonal series (not all points).

3°. Those which convert a single three-parameter (∞^3) family of isogonal series into isogonal series, and no other family of isogonal series of higher or equal number of parameters. Any transformation T of this class is such that it converts the ∞^2 points into points and such that it does not belong to the classes 1° or 2°.

4°. Those which convert a single four-parameter (∞^4) family of isogonal series into isogonal series, and no other family of higher or equal number of parameters. Any transformation of this class is such that it does not transform the ∞^2 points into points.

UNIONS INTO EQUI-TANGENTIAL SERIES.—THEOREM 3. *All element transformations are classified into four distinct classes with respect to the number of unions which become equi-tangential series.*

These classes are:

1°. Those which convert all (∞^∞) unions into equi-tangential series. Any transformation T of this group is the product of a contact transformation by a slide.

2°. Those which convert a single two-parameter (∞^2) family of unions into equi-tangential series, and no other family of unions of higher or equal number of parameters. Any transformation T of this class is characterized by the following three properties: (a) All straight lines are transformed into straight lines; (b) If C is the associated contact transformation, the distance d between the points of $T(E)$ and $C(E)$ is the same for all elements E on any line 1; (c) The distance d is not the same for all straight lines.

If any transformation T of this class is such that the distance d is the

same for all parallel straight lines, then the only unions which become equi-tangential series are the ∞^2 straight lines which are transformed into straight lines. If any transformation T of this class does not possess the preceding property, then there also exists a single one-parameter family of unions (not straight lines) which are transformed into equi-tangential series (not all unions).

3°. Those which convert two two-parameter ($2\infty^2$) families of unions into equi-tangential series, and no other family of unions of higher or equal number of parameters. These transformations are such that they transform ∞^2 unions into straight lines and such that they do not belong to the classes 1° or 2°.

Of course, one of the two-parameter families of unions which become equi-tangential series is the set of ∞^2 unions which are transformed into straight lines. The other two-parameter family of unions are carried into equi-tangential series which are not all unions.

4°. Those which convert a single three-parameter (∞^3) family of unions into equi-tangential series, and no other family of unions of higher or equal number of parameters. Any transformation T of this class is such that it transforms no set of ∞^2 unions into straight lines.

EQUI-TANGENTIAL SERIES INTO EQUI-TANGENTIAL SERIES. —THEOREM 4. *The element transformations are classified into five distinct classes with respect to the number of equi-tangential series which become equi-tangential series.*

These classes are defined as follows:

1°. Those which convert all (∞^∞) equi-tangential series into equi-tangential series. Any transformation T of this group must be the product of an equi-long transformation by a magnification by a slide.

2°. Those which convert a single two-parameter (∞^2) family of equi-tangential series into equi-tangential series, and no other family of equi-tangential series of higher or equal number of parameters. Any transformation T of this class is characterized by the following five properties: (a) All the straight lines are transformed into straight lines; (b) If C is the associated contact transformation, then under C the distance between the points of every two elements on any line l is magnified by the same quantity m; (c) The quantity m is the same for all parallel straight lines; (d) The distance between the points of $C(E)$ and $T(E)$ is the same for all parallel elements; (e) The transformation T does not belong to the class 1°.

For any transformation of this class, the only equi-tangential series which become equi-tangential series are the ∞^2 straight lines which are carried into straight lines.

3°. Those which convert two two-parameter ($2\infty^2$) families of equi-tangential series into equi-tangential series, and no other family of equi-tangential series of higher or equal number of parameters. Any trans-

formation T of this class is characterized by the following four properties: (a) All straight lines become straight lines; (b) If C is the associated contact transformation, then C converts all parallel straight lines into parallel straight lines; (c) The distance d between the points of $T(E)$ and $C(E)$ is the same for all elements E on any straight line 1; (d) The transformation T does not belong to the classes 1° or 2°.

One of the two-parameter families of equi-tangential series which are converted into equi-tangential series are the ∞^2 straight lines which are converted into straight lines. The other two-parameter family of equi-tangential series become equi-tangential series which are not unions.

4°. Those which convert a single three-parameter (∞^3) family of equi-tangential series into equi-tangential series, and no other family of equi-tangential series of higher or equal number of parameters. The transformations of this class are such that they carry the ∞^2 straight lines into straight lines and such that they do not belong to the classes 1°, 2° or 3°.

5°. Those which convert a single four-parameter (∞^4) family of equi-tangential series into equi-tangential series, and no other family of equi-tangential series of higher or equal number of parameters. Any transformation T of this class is such that it does not convert the ∞^2 straight lines into straight lines.

The first class in theorems 2 and 4 forms a group, and the intermediate classes obey the condition of involution (Poisson bracket vanishes). The possible sets are ∞^∞, ∞^4, ∞^3, $2\infty^2$, ∞^2. If, however, we allow for geometric degenerate duplications, the possibilities are ∞^∞, ∞^4, in theorems 2 and 4, and ∞^∞, ∞^3 in theorems 1 and 3.

ABSTRACT SELF-ADJOINT BOUNDARY CONDITIONS

By J. W. Calkin

The Institute for Advanced Study

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In studying the applications of the theory of Hilbert space to partial differential operators of the form

$$L = -\frac{\partial}{\partial x} p(x, y) \frac{\partial}{\partial x} - \frac{\partial}{\partial y} p(x, y) \frac{\partial}{\partial y} + q(x, y)\cdot,$$

we have considered the following problem: given a bounded set E in the (x, y)-plane with a smooth curve C as boundary, and the functions $p(x, y)$ and $q(x, y)$ defined and appropriately restricted on E, to characterize those boundary conditions on a function and its normal derivative at

C which define domains on which L is a maximal symmetric transformation or has a maximal symmetric closure in the Hilbert space $\mathfrak{L}_2(E)$. The nature of the results of the indicated investigations,[1] together with results in the theory of ordinary differential operators,[2] suggested strongly the possibility of formulating and discussing a problem in abstract terms, which would have as one realization that given above and as others, similar problems associated with a wide variety of differential operators both ordinary and partial. It is the purpose of the present note to formulate such a problem and to describe briefly the more salient features of its solution.

We employ, with minor modifications, the notation and terminology of Stone.[3] We make fundamental use of the theory of transformations between Hilbert and unitary spaces[4] and of the notion of the graph of a transformation.[5] The domain, range and graph of a transformation T will be denoted by $\mathfrak{D}(T)$, $\mathfrak{R}(T)$ and $\mathfrak{B}(T)$, respectively. We use the notation $T\mathfrak{N}$ to mean the set in the range of T into which T takes the set $\mathfrak{N}$ in its domain. Although we have occasion to consider simultaneously inner products formed in various spaces, we use the notation $(\ ,\ )$ for all such functions; the context indicates clearly in every case the space on which the product is formed.

DEFINITION 1. *Let H be a closed symmetric transformation in the Hilbert space $\mathfrak{H}$. A linear transformation A with domain in the graph $\mathfrak{B}(H^*)$ and range in a unitary or Hilbert space $\mathfrak{M}$ is called a reduction operator for H^* if it is closed, has domain dense in $\mathfrak{B}(H^*)$, range dense in $\mathfrak{M}$, and if the orthogonal complement in the space $\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{M}$ of $\mathfrak{B}(A)$ is the set of all vectors of the form $\{H^*f, -f, WA\{f, H^*f\}\}$, where W is a unitary transformation in $\mathfrak{M}$ such that $W^2 + I \equiv O$.*

It is readily proved that $\mathfrak{D}(A) \supseteq \mathfrak{B}(H)$ and that $A\{f, H^*f\} = 0$ if and only if f is in the domain of H. The case that H is self-adjoint and $A \equiv O$ is clearly trivial; hereafter we exclude it.

We denote by $\mathfrak{M}^+$ and $\mathfrak{M}^-$ the characteristic manifolds of W for the characteristic values i and $-i$, respectively. Evidently $\mathfrak{M}^- = \mathfrak{M} \ominus \mathfrak{M}^+$.

For simplification we write merely Af for $A\{f, H^*f\}$. Thus we have the abstract Lagrange identity

$$(f, H^*g) - (H^*f, g) + (Af, WAg) = 0 \qquad (1)$$

for all f and g for which Af and Ag are defined.

We describe now a general linear boundary condition associated with the transformation H^* of Definition 1.

DEFINITION 2. *If $\mathfrak{N}$ is a linear manifold in $\mathfrak{M}$, $\mathfrak{A}(\mathfrak{N})$ is the subset of elements f of $\mathfrak{D}(H^*)$ such that Af is defined and is in $\mathfrak{N}$. $H(\mathfrak{N})$ is the contraction of H^* with domain $\mathfrak{A}(\mathfrak{N})$. The requirement $Af \in \mathfrak{N}$ is called the boundary condition defining $H(\mathfrak{N})$. If the equation $Af = h$ does not have a*

solution in $\mathfrak{D}(H^*)$ for a dense set of elements h of $\mathfrak{N}$, the boundary condition $Af\epsilon\mathfrak{N}$ is said to be degenerate; otherwise it is said to be non-degenerate.

In order to characterize those extensions $H(\mathfrak{N})$ of H which are symmetric, we introduce

DEFINITION 3. *A linear manifold $\mathfrak{N}$ in $\mathfrak{M}$ is said to be W-symmetric if $W\mathfrak{N} \subseteq \mathfrak{M}\ominus\mathfrak{N}$. If $\mathfrak{N}$ has no proper W-symmetric extension, it is said to be maximal W-symmetric. If $W\mathfrak{N} = \mathfrak{M} \ominus \mathfrak{N}$, $\mathfrak{N}$ is said to be hypermaximal W-symmetric.*

It is clear from the formula (1) that the operator $H(\mathfrak{N})$ of Definition 2 is a symmetric extension of H if $\mathfrak{N}$ is W-symmetric and that every linear symmetric extension S of H such that $\mathfrak{B}(S) \subset \mathfrak{D}(A)$ can be characterized by a boundary condition $Af\epsilon\mathfrak{N}$ where $\mathfrak{N}$ is linear W-symmetric. For a complete discussion of the nature of the manifolds $\mathfrak{N}$ for which $H(\mathfrak{N})$ or $\bar{H}(\mathfrak{N})$ is maximal symmetric, the information provided by the following three theorems is essential.

THEOREM 1. *There is a one-to-one correspondence between the class of linear W-symmetric manifolds $\mathfrak{N}$ in $\mathfrak{M}$ and the class of all isometric transformations V with domains in $\mathfrak{M}^+$ and ranges in $\mathfrak{M}^-$; $\mathfrak{N}$ and V correspond if and only if $\mathfrak{N} = \mathfrak{R}(I - V)$. $\mathfrak{R}(I - V)$ is maximal W-symmetric if and only if $\mathfrak{D}(V) = \mathfrak{M}^+$ or $\mathfrak{R}(V) = \mathfrak{M}^-$, and hypermaximal W-symmetric if and only if both conditions are satisfied;*

$$\mathfrak{M} \ominus \mathfrak{R}(I - V) = W\mathfrak{R}(I - V)\oplus(\mathfrak{M}^+ \ominus \mathfrak{D}(V))\oplus(\mathfrak{M}^- \ominus \mathfrak{R}(V)).$$

THEOREM 2. *Let U be the transformation in $\mathfrak{H} \oplus \mathfrak{H}\oplus\mathfrak{M}$ which takes $\{f, g, h\}$ into $\{g, -f, Wh\}$. Then $U^2 + I \equiv 0$ and $\mathfrak{B}(A)$ is hypermaximal U-symmetric in $\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{M}$.*

THEOREM 3. *If $\mathfrak{M}$ has a finite dimension number or if A is bounded, the range of A is the entire space $\mathfrak{M}$.*

It is clear from Theorem 3 that, when $\mathfrak{M}$ is a unitary space or when A is bounded, no boundary condition $Af\epsilon\mathfrak{N}$ is degenerate and that every linear contraction of H^* is characterized by a boundary condition. Moreover it is readily proved by use of Theorem 3 that $H(\mathfrak{N})$ is maximal symmetric (self-adjoint) if and only if $\mathfrak{N}$ is maximal (hypermaximal) W-symmetric. A deeper analysis of the situation, involving study of the isometric operator associated with $\mathfrak{B}(A)$ by Theorems 2 and 1, leads to the following more precise result.

THEOREM 4. *If $\mathfrak{M}$ is a unitary space or A is bounded, there is a one-to-one correspondence between the class of all closed symmetric extensions S of H and the class of all closed W-symmetric manifolds $\mathfrak{N}$ in $\mathfrak{M}$; S and $\mathfrak{N}$ correspond if and only if $S \equiv H(\mathfrak{N})$ and $S^* \equiv H(\mathfrak{M} \ominus W\mathfrak{N})$. If $\mathfrak{N} = \mathfrak{R}(I - V)$, the deficiency-index of $H(\mathfrak{N})$ is (m, n), where m and n are the dimension numbers of the manifolds $\mathfrak{M}^- \ominus \mathfrak{R}(V)$ and $\mathfrak{M}^+ \ominus \mathfrak{D}(V)$, respectively. In particular, if j and k are the respective dimension numbers of $\mathfrak{M}^+$ and $\mathfrak{M}^-$, (k, j) is the deficiency-index of H.*

In the case that A is unbounded, examples can be constructed in which self-adjoint extensions S of H exist such that $\mathfrak{B}(S) \subset \mathfrak{D}(A)$, while neither $A\mathfrak{B}(S)$ nor its closure is maximal W-symmetric. On the other hand, we can also construct examples in which $\mathfrak{N}$ is a closed hypermaximal W-symmetric manifold and the equation $Af = h$ has a solution f in $\mathfrak{D}(H^*)$ for every h in $\mathfrak{N}$, while $H(\mathfrak{N})$ is not closed and $\tilde{H}(\mathfrak{N})$ is not maximal symmetric. Space limitations prevent us from discussing here our complete analysis of the situation in the case that A is unbounded; we shall merely state one theorem which to some extent typifies the sort of results that can be obtained. In the statement of the theorem, the symbol $\mathfrak{O} \oplus \mathfrak{O} \oplus \mathfrak{M}^+$ denotes the manifold of elements $\{0, 0, h_+\}$ of $\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{M}$ such that h_+ is in $\mathfrak{M}^+$.

THEOREM 5. *Let A be unbounded. Then $\mathfrak{M}^+$ and $\mathfrak{M}^-$ have each the dimension number $\aleph_0$ and the deficiency-index of H is $(\aleph_0, \aleph_0)$.*

Let Y be the isometric transformation between the characteristic manifolds of U associated with $\mathfrak{B}(A)$ by Theorems 2 and 1. Let $E_{\mathfrak{M}}$ be the projection in $\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{M}$ with range the set of vectors $\{0, 0, h\}$, where h is the general element of $\mathfrak{M}$. Then $E_{\mathfrak{M}} Y$ is defined everywhere in the manifold $\mathfrak{O} \oplus \mathfrak{O} \oplus \mathfrak{M}^+$. Let F be the transformation with domain $\mathfrak{M}^+$ such that $E_{\mathfrak{M}} Y\{0, 0, h_+\} = \{0, 0, Fh\}$. Then F is a bounded linear transformation from $\mathfrak{M}^+$ to $\mathfrak{M}^-$ and the adjoint F^* from $\mathfrak{M}^-$ to $\mathfrak{M}^+$ exists and has domain $\mathfrak{M}^-$.

Let V be an isometric transformation with domain in $\mathfrak{M}^+$ and range in $\mathfrak{M}^-$ and let $\mathfrak{N} = \mathfrak{N}(I - V)$. Then for $H(\mathfrak{N})$ to be maximal symmetric it is necessary and sufficient that

$$\text{either} \quad \mathfrak{N}(V - F) \supset A\mathfrak{B}(H(\mathfrak{M}^-)) \text{ or } \mathfrak{N}(V^{-1} - F^*) \supset A\mathfrak{B}(H(\mathfrak{M}^+)).$$

For $H(\mathfrak{N})$ to be self-adjoint it is necessary and sufficient that both conditions be satisfied.

The examples previously mentioned attest the significance of Theorem 5. We have also

THEOREM 6. *If A is unbounded, the class of maximal symmetric extensions S of H such that $\mathfrak{B}(S) \subset \mathfrak{D}(A)$ and the class of maximal symmetric extensions S of H such that $S \equiv \tilde{S}_0$, $\mathfrak{B}(S_0) \subset \mathfrak{D}(A)$, $\mathfrak{B}(S) \not\subset \mathfrak{D}(A)$, have each the cardinal number of the continuum.*

In conclusion, we point out that it is possible to define a reduction operator on the graph of the adjoint of an arbitrary closed symmetric transformation H in $\mathfrak{H}$. One has only to take as $\mathfrak{M}$ the space $\mathfrak{B}(H^*) \ominus \mathfrak{B}(H)$, as A the projection in $\mathfrak{B}(H^*)$ with range $\mathfrak{M}$.[6] Application of Theorem 4 to this situation leads again to the familiar method of constructing symmetric extensions of H by means of isometric extensions of its Cayley transform $X \equiv (H + iI)(H - iI)^{-1}$.[7]

The theory here described, together with related results and applications

to both ordinary and partial differential operators, will be discussed in detail in papers now being prepared for publication elsewhere.

[1] Some of the results are briefly described in *Bull. Amer. Math. Soc.*, abstracts 43–3–114, 43–3–209, 43–9–323. There are two errors of statement in the first abstract: on page 171, line 1, the symbols "$\subset$" should be replaced by "$\supseteq$" and on page 171, line 2, "H(W)" should be replaced by "$\tilde{\mathrm{H}}$(W)."

[2] M. H. Stone, *Linear Transformations in Hilbert Space*, New York, 1932, 424–530, especially Theorems 10.7 and 10.18. See also paper of I. Halperin, *Ann. Math.*, **38**, 880–919 (1937).

[3] *Opus cit.*

[4] For a discussion of the geometric aspects of the theory of transformations between Hilbert spaces, see F. J. Murray, *Trans. Amer. Math. Soc.*, **37**, 301–338 (1936), especially 301–312; the hypothesis that the transformations there considered have domains in a Hilbert space $\mathfrak{H}_1$ and range in a Hilbert space $\mathfrak{H}_2$ can be altered, in all of the theorems which we use, to allow either $\mathfrak{H}_1$ or $\mathfrak{H}_2$ or both to be unitary.

[5] J. v. Neumann, *Ann. Math.*, **33**, 294–310, especially 299 (1932); Murray, *loc. cit.*, 305–307.

[6] M. H. Stone has recently communicated to the author a characterization of all reduction operators which can be defined on the graph of the adjoint of a closed symmetric transformation H.

[7] J. v. Neumann, *Math. Ann.*, **102**, 49–131 (1929). See also Stone, *opus cit.*, Chap. IX.

A SIMPLIFIED METHOD FOR AUXIN EXTRACTION

BY J. VAN OVERBEEK

WILLIAM G. KERCKHOFF LABORATORIES, CALIFORNIA INSTITUTE OF TECHNOLOGY

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When the auxin content of a plant or part of it has to be determined two ways are open for analysis. (1) The oldest one is the *diffusion method*. The auxin of the plant material is allowed to diffuse into agar blocks. This method is more suited for determining the free moving auxin and the auxin production of tips and buds than for determining the actual auxin content of the plant. (2) The *extraction method* was first used by Thimann (1934). He extracted Avena seedlings by crushing them in chloroform and HCl. This procedure was repeated twice. Laibach and Meyer (1935) extracted maize and Helianthus with alcohol. The extract was concentrated and taken up in lanolin. This auxin paste was smeared on one side of intact Avena test plants. In corn plants up to 30 cm. in height no auxin could be found with exception of the coleoptile tips. Boysen Jensen (1937) reported good results with Thimann's method on Phaseolus. He also used ether and acetic acid and dropped the ether extract on agar

blocks (micromethod). Van Raalte (1937) extracted auxin from Vicia roots with ether and HCl.

When I tried to extract maize seedlings Thimann's chloroform and HCl method was first used. It failed to give results. Ether and acetic acid gave somewhat better results, but Boysen Jensen's dropping method could not be made to work satisfactorily. When acid was omitted from the extraction procedure, excellent results were obtained with ether extraction. Table 1 shows the increased yield from corn seedlings after omission of acid. It also shows that acid does not increase the amount of auxin extracted from Avena seedlings; in one case it even decreased it. The reason for the poor results with acid extraction may be the following. According to Kögl, Erxleben and Haagen Smit (1934) maize contains large amounts of auxin-b. This auxin is destroyed by acid. Another reason for the unsatisfactory preliminary results is the low auxin concentration of corn plants. After a suitable extraction method had been worked out it appeared that the auxin concentration of maize is about one-tenth of that of Avena. This was the more surprising since coleoptiles of corn produce large amounts of auxin in their tips.

The extraction method finally arrived at is described below. Its main features are the total omission of acid, the avoidance of crushing the material (considerable labor saving if large numbers of extractions have to be made), one single extraction for a prolonged time (overnight) and the evaporation of the ether extract to complete dryness.

a. Immediately before an extraction was made the ether was purified. Commercial ether was purified twice, recovered ether (see *c*) once. The ether was shaken with $FeSO_4$, CaO and water. In the beginning also $FeSO_4$, H_2SO_4 and water after Brandt (see Weisberger and Proskauer 1935) was used. Then the ether was distilled off. It was found that ether giving a negative benzidine test, a test which is considered extremely sensitive for detecting peroxides, still destroyed a hetero-auxin solution when shaken with it.

b. The plant material was measured (see *i*), cut into the parts that were to be analyzed and put uncrushed in Erlenmeyer flasks. Twenty-five to fifty cc. of ether was added per gram of plant material (less than 25 cc. was never used). This was left in a refrigerator for about 15 hours (overnight). If for some reason the material could not be tested after it had been in ether for 15 hours it was found advisable to evaporate the extract to dryness (see *c* and *d*) and store it in the refrigerator rather than to leave it in the ether. Material stored this way for three or four days had not lost in activity.

c. The ether was next separated from the plant material and from the water. Then it was distilled off (water bath 100°C.) until a residue of about 1 or $1^1/_2$ cc. was left. On account of the explosiveness of the ether

and for economical reasons, no amount larger than one of two cc. was allowed to evaporate in the open.

d. The residue was taken up in a pipet and dropped carefully on the bottom of a small vial (5 cc.) suspended in a beaker with boiling water. In this way the extract was evaporated to complete dryness and deposited at the bottom of the vial.

e. A known amount of agar (3%) was next pipetted on to the dry residue. Mostly 0.5 cc. was used. To secure a thorough mixing of agar and auxin the mixture was stirred and shaken vigorously (the vial still being in the boiling water). When the vial was removed from the boiling water it was left for a few hours for further uniform distribution of the auxin in the agar.

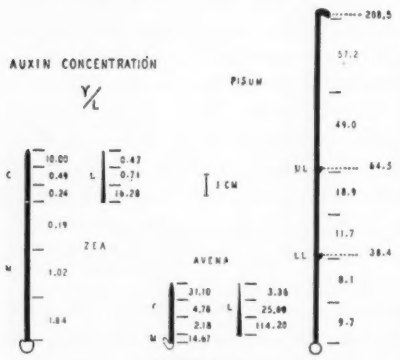

FIGURE 1

Auxin concentrations of etiolated seedlings of corn, oat and pea expressed in gammas hetero-auxin per liter water contained in the plant. It should be noticed that the auxin of the higher plants is of the auxin-a type which has an activity approximately twice that of hetero-auxin. In order to visualize the meaning of the figures it should be remembered that a hetero-auxin concentration of approximately 5 gammas per liter will give a curvature of one degree in the Avena test. Zea plants were 5 days, Avena 3 and Pisum 7 days old, which are stages of development at which these plants are mostly used in the laboratory. C, coleoptile. M, mesocotyl. L, primary leaf. UL, upper lateral bud. LL, lower lateral bud.

f. The agar auxin mixture was melted again in boiling water and poured into a rectangular brass mold (8 x 10.5 x 1.7 mm.). This mold was cooled by ice for a rapid gelification of the agar. Next this agar plate was divided into 12 equal blocks which were ready now to be put on decapitated oat seedlings for the auxin analysis. The extraction proper is thereby finished but in order to determine the auxin concentration in plants the following additional steps were taken:

g. The Avena test method described by Went and Thimann (1937, pp. 27–51) was employed. At the same time with the blocks to be analyzed a control series was run with a known concentration of hetero-auxin. This was necessary in order to calculate the obtained values in absolute amounts of auxin.

h. If weight determinations had to be made, the lengths of the stems, etc., were measured first, then the fresh weight was determined. Next the material was left for 24 hours in a drying oven, whereupon the water content could be determined.

i. The auxin concentration may be calculated from:

$$\frac{C_n \times I_{1^0} \times V_a}{W_n} \text{ gamma hetero-auxin equivalents per liter.}$$

C_n is the curvature of the Avena test for n plants per cm. coleoptile, mesocotyl, etc. I_{1^0} is the concentration of indole-3-acetic acid (hetero-auxin) in γ per liter required to give a curvature of 1° in the Avena test. V_a is the volume of agar in cc. in which the residue is taken up. W_n is the water content in grams per n plants per cm. of coleoptile, mesocotyl, etc.

TABLE 1

EFFECT OF ACID ON EXTRACTION. AUXIN DISTRIBUTION IN OAT SEEDLINGS GROWN IN WATER CULTURE AND IN SAND (70802, 70928)

WAY OF EXTRACTION	PLANT MATERIAL ANALYZED	AMOUNT OF AUXIN EXTRACTED IN DEGREES OF CURVATURE IN THE AVENA TEST	
Crushed and boiled with 50 cc. ether and 1 drop of 1 *N* acetic acid for 1 hour	14 corn seedlings	with acid	5.7
		acid omitted	13.3

Not crushed, left for 15 hours in 50 cc. ether and 0.5 cc. of 0.1 *N* acetic acid — 50 oat seedlings

Coleoptile (plus leaf)	AMOUNT OF AUXIN PER CM. ROOTS IN WATER ACID	ROOTS IN WATER NO ACID	ROOTS IN SAND NO ACID
Upper third	7.7	7.7	7.1
Middle third	2.9	2.7	3.0
Lower third	8.8	19.9	15.5
Mesocotyl			
Upper third			5.4
Middle third	14.0[1]	11.8[1]	9.3
Lower third			10.9

[1] Since the entire mesocotyl of water-grown plants was only 4 mm., the actual figures obtained in the assay were respectively 5.6 and 4.7.

In this way the auxin concentration is expressed in terms of hetero-auxin. In higher plants the auxin is of the auxin-a type, the activity of which is about twice that of hetero-auxin (according to Kögl, Haagen Smit and Erxleben).

Since the chemistry of auxin is known and hetero-auxin is readily available, it is my opinion that amounts of auxin extracted from or diffused out of plants should be expressed in absolute units rather than in arbitrary ones. The value of the arbitrary units (*A E*, *p u*, unit/cc., *W A E*) varies with the sensitivity of the test plants. An auxin amount expressed in hetero-auxin equivalents on the other hand is independent of the test method, test plant and sensitivity. However, in many cases where relative

amounts of auxin rather than absolute ones are being considered the auxin amounts may be more conveniently expressed in arbitrary units.

The above-described method has been used for the determination of the distribution of the auxin concentrations in maize, Avena and pea seedlings and also by Yin (unpublished as yet) to study the auxin distribution of Papaya leaves, and has proved to be reliable and to give reproducible results (Avena experiments of table 1). Figure 1 shows the auxin distribution in gammas hetero-auxin equivalents per liter. Note the high auxin concentration in the basal part of the primary leaves of Avena and maize and also the relatively high concentration in the basal parts of those seedlings. The concentration in the lateral buds of the pea seedling is higher than that in the adjacent stem tissue. A detailed paper discussing the relation between growth and the auxin distribution in maize and Avena seedlings and also the relation between auxin concentration and bud inhibition in the pea seedlings will be submitted to *The Botanical Gazette*.

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*LIGHT INTENSITY AND THE NITROGEN HUNGER PERIOD IN THE MANCHU SOYBEAN**

By E. B. Fred, P. W. Wilson and Orville Wyss

Departments of Agricultural Bacteriology and Agricultural Chemistry, University of Wisconsin

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Leguminous plants grown on a nitrogen-poor substrate and dependent on the fixation of atmospheric nitrogen for their supply of this element frequently exhibit during their development a "period of nitrogen hunger." This period occurs fairly early in the growth of the plant, when the stores of nitrogen in the seed have been exhausted and before the centers of fixation, the nodules, have developed sufficiently to meet the ever-increasing demands of the plant for nitrogen. As would be expected the phenomenon usually occurs under environmental conditions which favor photosynthesis,

and the usual evidence of carbohydrate excess are apparent. The plants are stalky with yellow leaves and woody tissues. Ordinarily the period lasts for less than a week, after which the nitrogen fixation process is initiated at a rate that adequately supplies the requirements of the plant for nitrogen. The leaves turn green, and the tissues become more succulent. Thereafter, the rate of carbohydrate synthesis rather than that of nitrogen fixation may become the limiting factor in the development of the plant.

In the summer of 1932 studies were begun at this station for the purpose of comparing the nitrogen metabolism of soybeans which were fixing nitrogen with that of plants which were supplied combined nitrogen. The first experiment was started out-of-doors early in June, just as the prolonged drought of that summer began. The weather during the period immediately following the planting of the soybeans was characterized by sunlight of high intensity and by hot, dry winds. The response of the nodulated soybeans to these rather extreme conditions was most unexpected—they entered the nitrogen hunger period and remained there. Plants supplied NH_4NO_3 developed normally. This difference in the response of the soybean plants to the environmental conditions suggested that the effect was concerned specifically with the nitrogen fixation process. Since the roots of the plants suffering from nitrogen hunger possessed numerous and well-developed nodules, it was concluded that the effect was primarily on the actual fixation of nitrogen. It occurred to us that perhaps the carbohydrate-nitrogen balance in the plant had become so excessive that assimilation of *free* nitrogen was inhibited. To test this hypothesis, part of the nodulated plants which were in the nitrogen hunger stage were shaded for one week in order to decrease carbohydrate formation and to increase the soluble forms of nitrogen in the plant. The response was clear-cut; in a few days nitrogen fixation had begun in the shaded plants, and at the end of a week, the leaves of these plants had become dark green. Analyses for nitrogen confirmed the observation that the shaded plants were markedly superior to the unshaded controls.

As has been indicated in our previous reports,[1,2,3] these results have important implications for several aspects of the mechanism of symbiotic nitrogen fixation. Among these may be mentioned: (a) the influence of the carbohydrate-nitrogen balance in the host plant on the process, and (b) the relative efficiency of free and combined nitrogen in the nutrition of the soybean. Because of these implications confirmation and extension of the observations are desirable. It should be noted that repetition of this type of work is not entirely in the hands of the experimenter as the requisite first stage, the inhibition of the fixation process, cannot be readily controlled. The weather must be *consistently* "bright and hot" during the first few weeks after planting in order that the nitrogen hunger stage will be prolonged to a point where inhibition of fixation will obtain. If at the

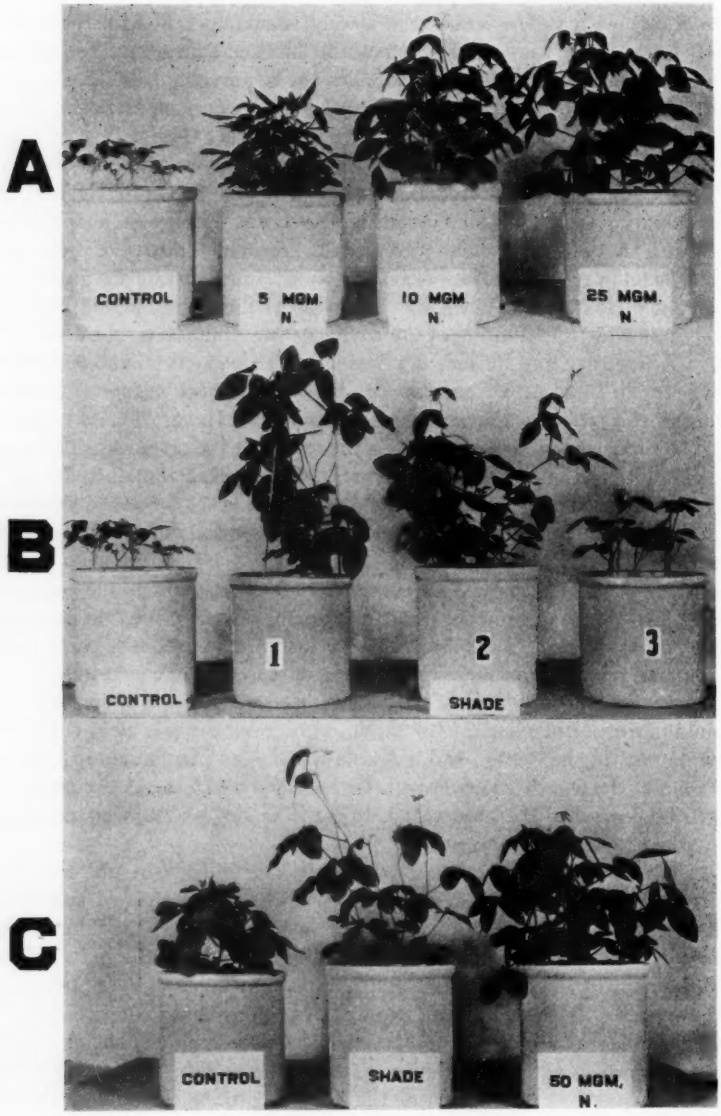

FIGURE 1

(See facing page)

critical stage in the growth of the plant, namely, when the nitrogen in the seed has been utilized, there are a few cool, cloudy days, nitrogen fixation will be initiated, and the further development of the plants is normal. Attempts to duplicate the phenomenon under the more easily controlled environment of the greenhouse with artificial illumination have met with little success, probably because of the low intensity of light available.

To increase the probability of securing plants in which the process of nitrogen fixation has been inhibited, experiments were made during the past two summers in which several jars of soybeans were planted every two weeks during June, July and early August. In 1936 inhibition was observed on two separate occasions, and this inhibition was overcome not only by shading as in our earlier experiments,[1] but also by addition of combined nitrogen. Support was thus obtained for the hypothesis that the inhibition was connected with an excessive carbohydrate-nitrogen relation in the plant. Unfortunately, no analyses were made of these plants, but the experiments were duplicated in 1937 as described in this paper.

Methods.—The methods used were those described previously.[1] Briefly, they consist of growing Manchu soybeans inoculated with an efficient strain of *Rhizobium japonicum* in two-gallon jars on a nitrogen-poor pit sand to which has been added an adequate supply of all plant nutrients except nitrogen. The plants are kept out-of-doors, protected from rain whenever necessary, and are watered daily with nitrogen-free tap water.

Experiment I.—On July 1, 1937, eight jars were planted and inoculated. The first signs of nitrogen hunger were noted about July 15; on July 28 the plants were still in the nitrogen hunger stage and were exhibiting pronounced signs of carbohydrate excess. On this date treatment was begun as follows: two jars were retained as controls; two jars were removed to a shaded cold-frame in which the light intensity was about one-fifth that of the cold-frame in the open; combined nitrogen was added to the remaining six jars. The plants which were shaded or to which combined nitrogen was added soon responded to these treatments. At the end of a week, their

EXPLANATION OF FIGURE

Effect of shading and of combined nitrogen on nodulated soybeans in which an excessive carbohydrate-nitrogen balance had inhibited the nitrogen fixation process.

A—Experiment I. Control: no treatment; to others combined nitrogen added, as $Ca(NO_3)_2$, 33 days before harvest.

B—Experiment I.
- Control: no treatment
- 1: in shade continuously for 33 days before harvest.
- 2: in shade for 12 days, returned to sun for 21 days.
- 3: in shade continuously for 15 days before harvest.

C—Experiment II.
- Control: no treatment
- Shade: in shade continuously for 19 days before harvest.
- 50 mg. N: added as $Ca(NO_3)_2$, 19 days before harvest.

color had changed from yellow to green, and they had increased in size. After 12 days one of the jars that had been kept in the shade was returned to the sun, inasmuch as the plants were showing the ill effects caused by low light intensity. The two controls in the sun remained yellow and showed no increase in size; on August 15 one of these controls was transferred to the shade. The response of the plants of this control was much slower than that of the first plants transferred, but eventualy they also began to turn green and were definitely superior in general appearance to those of the remaining control at harvest on August 30.

Experiment II.—Six jars were planted and inoculated on July 10, 1937. Nitrogen hunger was first apparent at the end of July and continued until

TABLE 1

EFFECT OF SHADING AND OF ADDING COMBINED NITROGEN ON NITROGEN FIXATION BY NODULATED SOYBEANS SUFFERING FROM A PROLONGED NITROGEN HUNGER PERIOD

TREATMENT	NUMBER OF PLANTS	DRY WEIGHT GM.	NITROGEN		
			PER CENT	TOTAL MGM.	FIXED PER PLANT MGM.
Experiment I					
Nodulated control kept in sun	5	4.2	1.14	47.9	2.6
Same + 5 mgm. N as $Ca(NO_3)_2$	6	20.7	1.26	260.5	35.6
Same + 10 mgm. N as $Ca(NO_3)_2$	7	28.5	1.28	364.8	43.7
Same + 25 mgm. N as $Ca(NO_3)_2$	6	30.8	1.45	446.9	63.3
Same + 50 mgm. N as $Ca(NO_3)_2$	8	37.7	1.72	648.1	67.7
In shade continuously for 33 days	6	8.9	2.20	195.8	25.6
In shade for 12 days, returned to sun	6	18.7	1.98	370.0	54.7
In shade for 15 days before harvest	6	6.4	1.08	69.2	4.5
Experiment II					
Nodulated control kept in sun	7	9.0	1.00	90.0	5.9
Same + 50 mgm. N as $Ca(NO_3)_2$	7	20.0	1.60	320.0	31.6
In shade continuously for 19 days	8	18.0	1.21	217.8	20.2

August 11, at which time treatments were started. Two jars were removed to the shade; 50 mgm. nitrogen as $Ca(NO_3)_2$ was added to two other jars; and the two remaining jars were retained as controls. These plants were harvested with those of Experiment I on August 30. Since duplicates of the same treatments were identical so far as general appearance was concerned, the plants of only one jar of each treatment were analyzed and the remaining jar after photographing, was used in other work.

Discussion.—Analytical data from the two experiments are summarized in table 1, and the appearance of the plants at harvest is shown in figure 1. The data of table 1 confirm and extend the observations previously reported. They demonstrate that it is unnecessary to remove the plants

from the high light intensity in order to initiate the nitrogen fixation process. Thus they lend support to the hypothesis that an excessive carbohydrate balance in the plant is the inhibitory factor rather than light itself. Furthermore, the breaking of the nitrogen hunger period by use of combined nitrogen indicates that the effect did not arise as a result of lowering the temperature concurrently with the reduction in light intensity. However, the excessive carbohydrate condition which apparently inhibits the nitrogen fixation process may arise in part from high temperatures, as well as from high intensity of light, since it is usually encountered only after periods of hot dry weather. Examination of the nodules of the plants kept in the sun confirmed the observation previously made,[1] namely, that the nodules were well developed and in no way resembled these on plants inoculated with a poor strain of bacteria.

Another point of interest is the difference between the controls (nodulated plants kept in the light with no addition of combined nitrogen) in the two experiments. At the harvest the control plants of Experiment II, although 10 days younger than those of Experiment I, were definitely larger and, in general appearance, were superior. Although the plants of the control jars of Experiment II were still quite yellow in color when harvested, there were some indications that the nitrogen hunger period was being broken by the cloudy weather just preceding the harvest. In contrast the control plants of Experiment I showed no signs of coming out of the nitrogen hunger stage even though they had been exposed to the same weather conditions. The quantitative difference in the behavior of the controls in the two experiments suggests that if the carbohydrate excess is not corrected early, it becomes increasingly difficult to overcome the inhibition. This view received confirmation by the response of those control plants in Experiment I which were transferred to the shade 15 days prior to the harvest. Although at harvest there were definite signs that nitrogen fixation had started in these plants, the inhibition was overcome much more slowly than in the plants which were moved to the shade soon after the nitrogen hunger period had begun.

As previously noted,[1,3] the *percentage nitrogen* in the plants is only a crude measure of the effective carbohydrate-nitrogen relation in these experiments. Both the plants which were shaded for 15 days in Experiment I and the control plants of Experiment II had a lower percentage nitrogen than did the control plants of Experiment I, in spite of the fact that the development of the latter plants was markedly inferior. The other treatments used, however, caused a definite reduction in the carbohydrate-nitrogen relation, as measured by an increase in the *percentage nitrogen*, coincident with overcoming the inhibition of the nitrogen fixation process.

Shading is extremely effective in breaking the nitrogen hunger period induced by excessive carbohydrates, but if the plants are kept in the shade,

they eventually become fragile and twiny, the usual symptoms in the soybean of inadequate illumination. Once the inhibition is overcome, reduced carbohydrate synthesis causes decreased fixation of nitrogen. If plants are returned to full sunlight after initiation of nitrogen fixation through shading development is normal, and excellent growth and fixation are obtained.

Finally, the quantitative aspects of the combined nitrogen treatments should be emphasized. The greatest fixation of elemental nitrogen was obtained with those plants supplied the largest quantities of combined nitrogen, a rather unusual finding, since combined nitrogen is commonly supposed to inhibit the fixation process.

In our other communications we have emphasized primarily the significance of these findings for theoretical phases of symbiotic nitrogen fixation, but their practical application to problems of agriculture should not be overlooked. Not infrequently experiment stations receive reports of crop failures with soybeans and other leguminous plants which are most puzzling, since apparently all cultural practices known to influence the crop yield, e.g., inoculation of the seed with bacteria of known efficiency, have been properly performed. It is suggested that leguminous plants, when seeded in late spring or early summer, may encounter local environmental conditions which will cause an excessive carbohydrate balance in the host plant and result in inhibition of nitrogen fixation. Our experiments indicate that all the necessary conditions will obtain only on soils of low fertility, expecially low with respect to presence of soluble forms of combined nitrogen. Investigation of local weather conditions after seeding of crop may throw light on some of these apparently inexplicable failures.

Summary. When nodulated soybeans are grown under sunlight of high intensity, fixation of atmospheric nitrogen is inhibited, and the nitrogen hunger stage in the plants is unduly prolonged. The inhibition appears to be associated with an excessive carbohydrate-nitrogen balance in the plant, probably with the relation of soluble forms of carbohydrate and nitrogen. Reduction of this excessive carbohydrate-nitrogen relationship either by shading (decrease in photosynthesis and hydrolysis of protein) or by addition of soluble forms of combined nitrogen is accompanied by initiation of the nitrogen fixation process, followed by a normal development of the plant.

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